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Theory and Simulation of Interacting Particle Systems

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Επιβλέπων:
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Διπλωματικη Εργασια

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Ε.Μ.Π.

Euqaristθec

Αρχικά, θα ήθελα να ευχαριστήσω ιδιαίτερα τον κ. Μιχαήλ Λουλάκη για τη συνεχή βοήθεια και καθοδήγηση του τον τελευταίο χρόνο. Η διεκπαιρέωση της παρούσας εργασίας ήταν εξαιρετικά ενδιαφέρουσα και δημιουργική.

Επιπλέον, θα ήθελα να ευχαριστήσω τους συμφοιτητές μου, οι οποίοι έκαναν την εκπόνηση των σπουδών μου στο Εθνικό Μετσόβιο Πολυτεχνείο πολύ ευχάριστη και ιδιαίτερα παραγωγική.

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Perílhyh

Τα Συστήματα Αλληλεπιδρώντων Σωματιδίων είναι μοντέλα που συναντώνται σε πολλά φυσικά συστήματα. Ο κύριος σκοπός της μελέτης τους είναι η εξαγωγή της μακροσκοπικής συμπεριφοράς από τη μικροσκοπική δυναμική. Στο φάσμα αυτής της εργασίας θεωρούμε πρώτα ένα σύστημα σωματιδίων χωρίς αλληλεπιδράσεις και συνάγουμε το μακροσκοπικό του προφίλ χρησιμοποιώντας τεχνικές που επεκτείνονται και σε συστήματα με αλληλεπιδράσεις. Στη συνέχεια, χρησιμοποιώντας αυτές τις τεχνικές, μελετάμε δύο από τα πιο δημοφιλή μοντέλα αλληλεπιδρώντων σωματιδιακών συστημάτων: Απλή Διαδικασία Αποκλεισμού και Διαδικασία Μηδενικού Εύρους. Επιπλέον, συζητάμε για την πιθανότητα καθολικότητας της κατανομής Tracy-Widom που εμφανίζεται πολύ συχνά σε συστήματα με αλληλεπιδρώντα δομικά στοιχεία. Τέλος, παρουσιάζουμε τα αποτελέσματα που προέκυψαν από τις προσομοιώσεις Monte Carlo στα Συστήματα Αλληλεπιδρώντων Σωματιδίων που μελετήσαμε.

Lèxeic Kleidiˆ

Markov διαδικασίες, συστήματα αλληλεπιδρώντων σωματιδίων, απλές διαδικασίες αποκλεισμού, διαδικασίες μηδενικού εύρους, μέθοδοι Monte Carlo

Abstract

Interacting Particle Systems are models encountered in many natural systems. The main purpose in their study is to deduce the macroscopic behavior from the microscopic dynamics. In the spectre of this thesis we consider first a particle system without interaction and deduce its macroscopic profile using techniques which are also applied in systems with interaction. Then, using these techniques, we study two of the most popular models of Interacting Particle Systems: Simple Exclusion Process and Zero Range Process. In addition, we discuss the possible universality of the Tracy-Widom distribution which very often emerges in systems with interacting components. Last but not least, we present results acquired by Monte Carlo simulations on our Interacting Particle Systems.

Keywords

Markov Processes, Interacting Particle Systems, Simple Exclusion Processes, Zero Range Processes, Monte Carlo Methods

Contents

Chapter 1

Introduction

Interacting Particle Systems are models encountered in many natural systems, for which the local mechanics are very simple, but it might be considerably difficult to extract a global behavior. In most cases, the factors that contribute to this difficulty is the introduction of stochastic dynamics and interaction into them. Examples can be found in problems from natural sciences, such as reaction diffusion and gas particles systems, extending to problems from social sciences such as traffic flow, opinion dynamics and spread of epidemics.

The main purpose in the study of such Interacting Particle Systems is to deduce the macroscopic behavior, which is usually described by hydrodynamic equations, from the microscopic interaction, namely the underlying stochastics. The mathematical term associated to this is called the *scaling limits*. Scaling limits are of great interest in physics and in particular in mathematics. Furthermore, it usually interests us to find equilibrium states in our system, namely when the macroscopic profile does not change. These equilibriums are described mathematically by the establishment of *invariant distributions*.

1.1 Object of Thesis

On the one part, the goal of the current thesis is to study and apply mathematical techniques on the subject of Interacting Particle Systems and help us elucidate specific behaviors from a theoretical view. In this study we will consider the following models of Particle Systems:

Independent Random Walks

Simple Exclusion Process

Zero Range Process

On the other part, we present Monte Carlo simulations on such models, which help us visualize their characteristics and approximate specific scaling limits for which there has not been an analytic result.

1.2 Following Chapters

In chapter 2 we present the basic concepts and "tools" from the field of Stochastic Processes which will assist us to establish concepts in Interacting Particle Systems.
In chapter 3 we will attempt to make the reader familiar with the mathematical techniques usually applied on Interacting Particle Systems.
In chapter 4 we will study Simple Exclusion Process and establish an invariant distribution. Moreover, we will give results on a certain variance of the process and discuss the emergence of a special distribution.
In chapter 5 we will study Zero Range Process, establish again an invariant distribution and discuss special properties of the model.
In chapter 6 we will present the results acquired by the Monte Carlo simulations on our Interacting Particle Systems.

Chapter 2

Theoretical Background

In this chapter we will introduce basic theorems and definitions of Stochastic Processes that will be used on a regular basis in the following chapters. Keep in mind that each Interacting Particle System is a continuous-time Markov process describing the collective behavior of stochastically interacting components. We use this section to state properties related to Markov processes which yield the background for our following investigation on interacting particle systems.

2.1 Markov Processes

Let us start with some basic notation. For a topological space X, we denote with $\mathcal{B}(X)$ the Borel σ-algebra generated by the open sets of X and equip X with this σ-algebra if not stated otherwise. $\mathcal{M}(X)$ denotes the space of Borel measures on X and $\mathcal{M}_1(X)$ is the subset of all probability measures.

A continuous time stochastic process $(\eta_t)_{t \geq 0}$ is a family of random variables η_t taking values in a compact metric space X, which is called the state space of the process. Let

$$D[0, \infty) = \{\eta : [0, \infty) \to X \text{ cadlagg}\}$$

be the set of right continuous functions with left limits (*cadlag*), which is the canonical path space for a stochastic process on X. To define a reasonable measurable structure on $D[0, \infty)$, namely a suitable σ-algebra, let $\mathcal{F}$ be the smallest σ-algebra on $D[0, \infty)$ such that all the mappings $\eta \mapsto \eta_s$ for $s \geq 0$ are measurable with respect to $\mathcal{F}$. That means that every path can be evaluated at arbitrary times s, namely

$$\{\eta_s \in A\} = \{\eta \mid \eta_s \in A\} \in \mathcal{F}$$

for all measurable subsets $A \in X$. If $\mathcal{F}_t$ is the smallest σ-algebra on $D[0, \infty)$ relative to which all the mappings $\eta \mapsto \eta_s$ for $s \leq t$ are measurable, then

$(\mathcal{F}_t : t \geq 0)$ provides a natural filtration for the process. The filtered space $\{D[0,\infty), \mathcal{F}, (\mathcal{F}_t : t \geq 0)\}$ provides a generic choice for the probability space of a stochastic process which can be defined as a probability measure $\mathbb{P}$ on $D[0,\infty)$.

Definition 1. A (time-homogeneous) *Markov process* on X is a collection $(\mathbb{P}^\zeta : \zeta \in X)$ of probability measures on $D[0,\infty)$ with the following properties:

1. $\mathbb{P}^\zeta(\eta \in D[0,\infty) : \eta_0 = \zeta) = 1$ for all $\zeta \in X$, namely $\mathbb{P}^\zeta$ is normalized on all paths with initial condition $\eta_0 = \zeta$.
2. The mapping $\zeta \mapsto \mathbb{P}^\zeta(A)$ is measurable for every $A \in \mathcal{F}$.
3. $\mathbb{P}^\zeta(\eta_{t+\cdot} \in A|\mathcal{F}_t) = \mathbb{P}^{\eta_t}(A)$ for all $\zeta \in X$, $A \in \mathcal{F}$ and $t > 0$. (Markov property)

2.2 Markov Chains

Let X now be a countable set. Then a Markov process $(\eta_t)_{t\geq 0}$ is called a *Markov chain* and it can be characterized by *transition rates* $c(\eta,\eta') \geq 0$, which have to be specified for all $\eta,\eta' \in X$. Often $c(\cdot,\cdot)$ is described as a matrix. For a given process $(\mathbb{P}^\eta : \eta \in X)$ the rates are defined via

$$\mathbb{P}^\eta(\eta_t = \eta') = c(\eta,\eta')t + o(t) \text{ as } t \searrow 0, \tag{2.1}$$

and represent probabilities per unit time.

At this point, we would like to give an intuitive understanding of the time evolution and the role of the transition rates in a process. Denote by

$$W_\eta := \inf\{t \geq 0 : \eta_t \neq \eta\}$$

the *holding time* in state η. The value of this time is related to the total exit rate out of state η,

$$c_\eta := \sum_{\eta' \neq \eta} c(\eta,\eta').$$

If $c_\eta = 0$, η is called an *absorbing state* and $W_\eta = \infty$.

Proposition 1. *If* $c_\eta > 0$*, then* $W_\eta \sim Exp(c_\eta)$ *and* $\mathbb{P}^\eta(\eta_{W_\eta} = \eta') = c(\eta,\eta')/c_\eta$.

Proof. W_η has the Markov property

$$\mathbb{P}^\eta(W_\eta > s+t|W_\eta > s) = \mathbb{P}^\eta(W_\eta > s+t|\eta_s = \eta) = \mathbb{P}^\eta(W_\eta > t).$$

Therefore $\mathbb{P}^\eta(W_\eta > s+t) = \mathbb{P}^\eta(W_\eta > s)\mathbb{P}^\eta(W_\eta > t)$. This is the functional equation for an exponential and it suggests that

$$\mathbb{P}^\eta(W_\eta > t) = e^{-c_\eta t} \text{ with initial condition } \mathbb{P}^\eta(W_\eta > 0) = 1.$$

For the parameter λ we will have that

$$\lambda = -\frac{d}{dt}P^{\eta}(W > t)\Big|_{t=0} = \lim_{t \searrow 0}\frac{P^{\eta}(W > t) - 1}{t} = -c_{\eta} \, ;$$

since according to equation 2.1

$$P^{\eta}(W > 0) = 1 - P^{\eta}(\eta_t \neq \eta) + o(t) = 1 - c_{\eta} t + o(t).$$

Now, conditioned on a jump occurring we get

$$P^{\eta}(\eta_t = \eta' | W < t) = \frac{P^{\eta}(\eta_t = \eta')}{P^{\eta}(W < t)} \to \frac{c(\eta, \eta')}{c_{\eta}} \text{ as } t \searrow 0$$

by L' Hospital's rule. With the right-continuity of paths, this implies the second statement. □

Remark 1. Let $\{W_1, \dots, W_n\}$ be a sequence of independent exponentials $W_i \sim Exp(\lambda_i)$. Regarding the distribution of $W = \min\{W_1, \dots, W_n\}$, we will have that

$$P(W > t) = P(W_1 > t, \dots, W_n > t) =$$
$$P(W_1 > t) \cdots P(W_n > t) = e^{-\lambda_1 t} \cdots e^{-\lambda_n t} = e^{-(\lambda_1 + \cdots + \lambda_n)t}$$
$$\Leftrightarrow W \sim Exp\Big(\sum_{i=1}^{n} \lambda_i\Big).$$

Keep in mind that what was presented in this section, will be very useful at the design of our simulations in chapter 6.

2.3 Feller Processes, Semigroups and Generators

Let X be a compact metric space and denote by

$$C(X) = \{f : X \to \mathbb{R} \; continuous\}$$

the set of real-valued continuous functions, which is a Banach space with sup-norm $\|f\|_{\infty} = \sup_{\eta \in X}|f(\eta)|$. Functions f can be regarded as observables, and we are interested in their time evolution.

Definition 2. For a given process $(\eta_t)_{t \geq 0}$ on X, for each $t \geq 0$ we define the operator $S(t) : C(X) \to C(X)$ by

$$S(t)f(\eta) := \mathbb{E}^{\eta} f(\eta_t).$$

In general $f \in C(X)$ does not imply $S(t)f \in C(X)$, but all the processes we consider have this property and are called *Feller processes*.

Proposition 2. *Let η be a Feller process on X. Then the family $(S(t) : t \geq 0)$ is a Markov semigroup, namely*

1. $S(0) = Id$, *(identity at $t = 0$)*
2. $t \mapsto S(t)f$ *is right-continuous for all $f \in C(X)$, (right-continuity)*
3. $S(t+s)f = S(t)S(s)f$ *for all $f \in C(X)$, $s, t \geq 0$, (Markov property)*
4. $S(t)1 = 1$ *for all $t \geq 0$, (conservation of probability)*
5. $S(t)f \geq 0$ *for all non-negative $f \in C(X)$. (positivity)*

Proof. 1. $S(0)f(\eta) = \mathbb{E}^\eta(f(\eta_0)) = f(\eta)$ since $\eta_0 = \eta$ (def. 1(1)).

2. follows from right-continuity of η_t and continuity of f.

3. follows from the Markov property of η_t (def. 1(3))

$$S(t+s)f(\eta) =$$
$$\mathbb{E}^\eta f(\eta_{t+s}) = \mathbb{E}^\eta(\mathbb{E}(f(\eta_{t+s})|\mathcal{F}_t)) = \mathbb{E}^\eta(\mathbb{E}^{\eta_t}(f(\eta_{t+s}))) = \mathbb{E}^\eta((S(s)f)(\eta_t))$$
$$= S(t)S(s)f.$$

4. $S(t)1 = \mathbb{E}^\eta(1_{\eta_t}(X)) = 1$ since $\eta_t \in X$ for all $t \geq 0$ (conservation of probability).

5. is immediate by definition.

□

The Markov semigroup $S(t)$ will appear frequently in our computations and theorems as it expresses in a natural way how a process will "behave" at a later time t. It determines the expected value of observables f on X at time t for a given Markov process η. Specification of all these expected values provides a full representation of η.

Let $P(X)$ denote the set of all probability measures on X, with the topology of *weak convergence*:

$$\mu_n \to \mu \quad \text{if and only if} \quad \int f \, d\mu_n \to \int f \, d\mu$$

for all $f \in C(X)$. Note in particular that with respect to this topology, $P(X)$ is compact since X is compact. If $\mu \in P(X)$ and $\{\mathbb{P}^\eta ; \eta \in X\}$ is a Markov process, then the corresponding Markov process with initial distribution μ is a stochastic process η_t whose distribution is given by

$$\mathbb{P}^\mu := \int_X \mathbb{P}^\eta \, \mu(d\eta).$$

In view of this,

$$\mathbb{E}\, f(\eta_t) = \int_X S(t) f \, d\mu$$

for all $f \in C(X)$. This leads to the following definition.

Definition 3. For a process $(\eta_t : t \geq 0)$ with initial distribution μ we denote by $\mu S(t) \in P(X)$ the *distribution at time* t, which is uniquely determined by

$$\int_X f \, d[\mu S(t)] := \int_X S(t) f \, d\mu$$

for all $f \in C(X)$.

Now, since $(S(t) : t \geq 0)$ has the Markov property, in analogy with the proof of proposition 1 we expect that it has the form of an exponential generated by the linearization $S'(0)$, namely

$$S(t) = \text{"}e^{S'(0)t}\text{"} = Id + S'(0)t + o(t) \text{ with } S(0) = Id;$$

which is made precise in the following.

Definition 4. The generator $L : D_L \to C(X)$ for the process $(S(t) : t \geq 0)$ is given by

$$Lf := \lim_{t \searrow 0} \frac{S(t)f - f}{t}$$

for $f \in D_L$, where the domain $D_L \subseteq C(X)$ is the set of functions for which the limit is exists.

Note that, in general, D_L is a proper subset of $C(X)$ for processes on infinite lattices, and this is in fact the case even for the simplest examples.

Proposition 3. *L as defined above is a Markov Generator, namely*

1. *$1 \in D_L$ and $L1 = 0$, (conservation of probability)*
2. *for $f \in D_L$, $\lambda \geq 0$: $\min_{\zeta \in X} f(\zeta) \geq \min_{\zeta \in X} (f - \lambda Lf)(\zeta)$, (positivity)*
3. *D_L is dense in $C(X)$ and the range $R(Id - \lambda L) = C(X)$ for sufficiently small $\lambda > 0$.*

The proof is rather technical and can be found in [12].

In general, for Markov chains with countable X and jump rates $c(\eta, \eta')$ the generator is given by

$$Lf(\eta) = \sum_{\eta' \in X} c(\eta, \eta')\big(f(\eta') - f(\eta)\big)$$

which, using equation 2.1, follows for small $t \searrow 0$ from

$$S(t)f(\eta) = \mathbb{E}^{\eta}\big(f(\eta_t)\big) = \sum_{\eta' \in X} \mathbb{P}^{\eta}(\eta_t = \eta')\, f(\eta')$$

$$= \sum_{\eta' \neq \eta} c(\eta, \eta') f(\eta')\, t + f(\eta) \Big(1 - \sum_{\eta' \neq \eta} c(\eta, \eta')\, t\Big) + o(t)$$

and the definition of L.

Definition 5. For $X = \{0,1\}^{\Lambda}$, where Λ is a countable lattice, $f \in C(X)$ is a *cylinder function* if there exists a finite subset $\Delta_f \subseteq \Lambda$ such that $f(\eta^x) = f(\eta)$ for all $x \notin \Delta_f$, $\eta \in X$, namely f depends only on a finite set of coordinates of a configuration. We write $C_0(X) \subseteq C(X)$ for the set of all cylinder functions.

2.4 Invariant Measures

One of the main questions we need to address in the study of Interacting Particle Systems is the characterization of all invariant measures.

Definition 6. A measure μ is *invariant* or *stationary* if $\mu S(t) = \mu$. Equivalently,

$$\int_X S(t) f \, d\mu = \int_X f \, d\mu$$

or shorter $\mu(S(t)f) = \mu(f)$ for all $f \in C(X)$.

The set of all invariant measures of a process is denoted by $\mathcal{I}$. In addition, a measure μ is called *reversible* if $\mu(fS(t)g) = \mu(gS(t)f)$ for all $f, g \in C(X)$.

Taking $g = 1$ in the previous equation we see that every reversible measure is also stationary. Stationarity of μ implies that

$$\mathbb{P}^{\mu}(\eta_{\cdot} \in A) = \mathbb{P}^{\mu}(\eta_{t+\cdot} \in A)$$

for all $t \geq 0, A \in \mathcal{F}$, namely if a state η follows a distribution with respect to μ, it will continue to do so after time t. Using $\eta_t \sim \mu$ as initial distribution, the definition of a stationary process can also be extended to negative times on the path space $D(-\infty; \infty)$.

Proposition 4. *Consider a Feller process on a compact state space X with generator L. Then*

$$\mu \in \mathcal{I} \iff \mu(Lf) = 0$$

for all $f \in C_0(X)$, and similarly

$$\mu \text{ is reversible} \iff \mu(fLg) = \mu(gLf)$$

for all $f, g \in C_0(X)$.

Proof. Follows from the definitions of semigroup/generator and the fact that $\mu(f_n) \to \mu(f)$ if $\|f_n - f\|_\infty \to 0$ by continuity of f_n, f and compactness of X. □

In particular, not every Markov chain has an invariant distribution. If X is finite there exists at least one invariant distribution, as a direct result of linear algebra (Perron-Frobenius theorem). For Interacting Particle Systems we have compact state spaces X, for which a similar result holds.

Theorem 1. *For every Feller process with compact state space X we have:*

1. *$\mathcal{I}$ is non-empty, compact and convex.*
2. *Suppose the weak limit $\mu = \lim_{t\to\infty} \nu S(t)$ exists for some initial distribution $\nu \in \mathcal{P}(X)$, namely*

$$\nu S(t)(f) = \int_X S(t)f \, d\nu \to \mu(f)$$

   for all $f \in C(X)$, then $\mu \in \mathcal{I}$.

For the proof see Theorem 1.9 in [6].

Definition 7. A Markov process $(\mathbb{P}^\eta : \eta \in X)$ is called *irreducible*, if for all $\eta, \eta' \in X$

$$\mathbb{P}^\eta(\eta_t = \eta') > 0$$

for some $t \geq 0$.

So an irreducible Markov process can sample the whole state space, and if X is countable this implies that it has at most one invariant distribution.

Definition 8. A Markov process with semigroup $(S(t) : t \geq 0)$ is *ergodic* if

1. $\mathcal{I} = \{\mu\}$ is a singleton, and (unique stationary measure)
2. and $\lim_{t\to\infty} \nu S(t) = \mu$ for all $\nu \in \mathcal{P}(X)$. (convergence to equilibrium)

Note that in an irreducible Markov process we can observe phase transitions, that is, mathematically speaking, a change between invariant distributions. Phase transitions are related to the breakdown of ergodicity in irreducible systems, in particular, non-uniqueness of invariant measures.

Proposition 5. *An irreducible Markov chain with finite state space X is ergodic.*

Proof. A result of linear algebra, in particular the Perron-Frobenius theorem: The finite matrix $c(\eta, \eta')$ has eigenvalue 0 with unique eigenvector μ. □

Consequently, mathematically phase transitions occur only in infinite systems. Infinite systems are often studied as limits of finite systems, which show traces of a phase transition by divergence or non-analytic behavior of certain observables. In terms of applications, infinite systems are approximations or idealizations of large finite systems, so results have to be interpreted with "care".

Chapter 3

Independent Random Walks

In this chapter we want to investigate the system of indistinguishable particles following independent random walks. Our main goal is for the reader to get familiar with the main concepts and techniques used in the field of Interacting Particles System.

3.1 Model

Denote by $\mathbb{Z}^d$ the d-dimensional integer lattice. For a positive integer L, denote by $\mathbb{T}_L$ the torus with L points: $\mathbb{T}_L = \mathbb{Z}_L = \{0, 1, \ldots, L-1\}$ and let $\mathbb{T}_L^d = (\mathbb{T}_L)^d$. Here L represents the inverse of the distance between the points of $\mathbb{T}_L^d$, namely the particle sites, which are represented by x, y and z. By letting $L \to \infty$, the distance between particles will go to zero and so, we pass from microscopic to macroscopic.

We want to describe the evolution of the system, so let N denote the total number of particles and let $x_1, x_2, \ldots x_N$ denote their initial positions. Also, because particles evolve as independent translation invariant discrete time random walks on the torus, we need to fix a translation invariant transition probability $p(x, y)$ on $\mathbb{Z}^d$, for which $p(x, y) = p(0, y-x) =: p(y-x)$ for some probability $p(\cdot)$ on $\mathbb{Z}^d$, called the *elementary transition probability* of the system. This probability expresses the stochastic characteristic of the random walk.

Let $p_t(x, y)$ represent the probability of being at time t on site y for a discrete time random walk with elementary transition probability $p(\cdot)$ starting from x. In addition, we have that $p_t(\cdot, \cdot)$ inherits the translation invariance property from $p(\cdot, \cdot)$, and so $p_t(x, y) = p_t(0, y-x) =: p_t(y-x)$.

It is time to describe the motion of each particle. Let us take N independent random walk variables $\{Z_t^1, Z_t^2, \ldots, Z_T^N\}$ on $\mathbb{Z}^d$ with elementary transition probability $p(\cdot)$ and initially at zero. As a result, the position of each particle i on the torus $\mathbb{T}_L^d$ at time t will be

$$X_t^i = x_i + Z_t^i \ mod\ L.$$

However, since particles are indistinguishable in our model, it does not interest us the exact position of each particle, but the number of particles on each site of the space. Specifically, the state space of the system, also called configuration space, is $\mathbb{N}^{\mathbb{T}_L^d}$. Configurations are denoted by η, ξ and ζ. Under this definition, if x is a site of $\mathbb{T}_L^d$, then $\eta(x)$ is the number of particles on this site for the configuration η. Therefore, if the particles are initially at $x_1, x_2, \ldots x_N$, then

$$\eta(x) = \sum_{i=1}^{N} \mathbf{1}\{x = x_i\}.$$

On the other hand, if we are given $(\eta(x) : x \in \mathbb{T}_L^d)$, we can first label the particles and then let them evolve according to the stochastic dynamics we have described.

Of course, we want the configuration at time t, which will be denoted by $\eta_t(x)$ and defined by

$$\eta_t(x) = \sum_{i=1}^{N} \mathbf{1}\{x = X_t^i\}.$$

Moreover,the process $(\eta_t)_{t \geq 0}$ inherits the Markov property from the random walks $\{X_t^i; 1 \leq i \leq N\}$ because all particles have the same elementary transition probability and they do not interact with each other.

3.2 Poisson Measure

Since the state space is finite and since the total number of particles is the unique quantity conserved by the dynamics of the system, for every positive integer N representing the total number of particles, there is only one invariant measure, as long as the support of the elementary transition probability $p(\cdot)$ generates $\mathbb{Z}^d$, namely the process is irreducible. The Poisson measures in our study will play a central role.

Recall that a Poisson distribution of parameter $\lambda > 0$ is the probability measure $\{p_{\lambda,k} = p_k; k > 1\}$ on $\mathbb{N}$ given by

$$p_k = e^{-\lambda}\frac{\lambda^k}{k!}, \; k \in \mathbb{N}$$

and its Laplace transform is equal to

$$e^{-\lambda}\sum_{k=0}^{\infty} e^{-\theta k}\frac{\lambda^k}{k!} = e^{-\lambda} e^{\lambda e^{-\theta}} = e^{\lambda(e^{-\theta}-1)},$$

for all $\theta > 0$.

Definition 9. For a fixed positive function $\rho : \mathbb{T}_L^d \to \mathbb{R}_+$, we call Poisson measure on $\mathbb{T}_L^d$ associated to the function ρ, a probability on the configuration space $\mathbb{N}^{\mathbb{T}_L^d}$, denoted by $\nu^L_{\rho(\cdot)}$, having the following two properties:

- under $\nu^L_{\rho(\cdot)}$ the random variables $(\eta(x) : x \in \mathbb{T}^d_L)$, representing the number of particles at each site, must be independent,

- for every fixed site $x \in \mathbb{T}^d_L$, $\eta(x)$ is distributed according to a Poisson distribution of parameter $\rho(x)$.

In the case where the function ρ is constant equal to ρ, we denote $\nu^L_{\rho(\cdot)}$, just by ν^L_ρ. We will, also, denote expectation with respect to a measure μ by E_μ.

The measure $\nu^L_{\rho(\cdot)}$ is characterized by its multidimensional Laplace transform:

$$E_{\nu^L_{\rho(\cdot)}}\left[e^{-\sum_{x\in\mathbb{T}^d_L}\lambda(x)\eta(x)}\right] = \prod_{x\in\mathbb{T}^d_L} e^{\rho(x)\left(e^{-\lambda(x)}-1\right)} = e^{\sum_{x\in\mathbb{T}^d_L}\rho(x)\left(e^{-\lambda(x)}-1\right)}$$

for all positive sequences $(\lambda(x) : x \in \mathbb{T}^d_L)$ [4].

Now, let us move on to establishing that the Poisson measures associated to constant functions are invariant for a system of independent random walks.

Proposition 6. *If particles are initially distributed according to a Poisson measure associated to a constant function equal to ρ then the distribution at time t is exactly the same Poisson measure. [11]*

Proof. Denote by $\mathbb{P}_{\nu^L_\rho}$ the probability measure on the path space $\Omega_L = \mathbb{N}^{\mathbb{T}^d_L} \times \cdots \times \mathbb{N}^{\mathbb{T}^d_L}$, namely the space of $(\eta_t)_{t\geq 0}$, induced by the independent random walk dynamics and the initial measure ν^L_ρ. Expectation with respect to $\mathbb{P}_{\nu^L_\rho}$ is denoted by $\mathbb{E}_{\nu^L_\rho}$. At this point, notice the difference between $E_{\nu^L_\rho}$, which is the expectation with respect to the measure defined on $\mathbb{N}^{\mathbb{T}^d_L}$, and $\mathbb{E}_{\nu^L_\rho}$, which is the expectation with respect to the measure defined on the path space Ω_L. It is easy to see that

$$\mathbb{E}_{\nu^L_\rho}[F(\eta_0)] = E_{\nu^L_\rho}[F(\eta)]$$

for all bounded continuous functions F on $\mathbb{N}^{\mathbb{T}^d_L}$.

Since the measure $\nu^L_{\rho(\cdot)}$ is characterized by its multidimensional Laplace transform, we will compute here the expectation

$$\mathbb{E}_{\nu^L_\rho}\left[e^{-\sum_{x\in\mathbb{T}^d_L}\lambda(x)\eta_t(x)}\right]$$

for all positive sequences $(\lambda(x) : x \in \mathbb{T}^d_L)$. Furthermore, for a site $y \in \mathbb{T}^d_L$, we will denote by $X^{y,k}_t$ the position at time t of the k-th particle starting from y. In this way, the number of particles on site x at time t will be:

$$\eta_t(x) = \sum_{y\in\mathbb{T}^d_L}\sum_{k=1}^{\eta_0(y)} \mathbb{1}\{x = X^{y,k}_t\}.$$

And then, by inverting the order of summation, we will get that:

$$\sum_{x\in\mathbb{T}_L^d} \phi(x)\,\eta_t(x) = \sum_{y\in\mathbb{T}_L^d}\sum_{k=1}^{\eta_0(y)} \phi(X_t^{y;k}).$$

Since each particle evolves independently and the total number of particles at each site at time 0 is distributed according to a Poisson distribution of parameter ρ,

$$\begin{aligned}
\mathbb{E}_{\mu_\rho^L}\left[e^{\sum_{x\in\mathbb{T}_L^d}\phi(x)\eta_t(x)}\right] &= \mathbb{E}_{\mu_\rho^L}\left[e^{\sum_{y\in\mathbb{T}_L^d}\sum_{k=1}^{\eta_0(y)}\phi(X_t^{y;k})}\right] = \\
&\prod_{y\in\mathbb{T}_L^d}\mathbb{E}_{\mu_\rho^L}\left[e^{\sum_{k=1}^{\eta_0(y)}\phi(X_t^{y;k})}\right] = \\
&\prod_{y\in\mathbb{T}_L^d}\int \mu_\rho^L(d\eta)\,E\left[e^{\phi(X_t^{y;1})}\right]^{\eta_0(y)} = \\
&\prod_{y\in\mathbb{T}_L^d} e^{\rho\left(E\left[e^{\phi(X_t^{y;1})}\right]-1\right)} = \prod_{y\in\mathbb{T}_L^d} e^{\rho\left(E\left[e^{\phi(y+X_t)}\right]-1\right)}
\end{aligned}$$

where X_t is a random walk at time t on the torus $\mathbb{T}_L^d$ starting from the origin and with transition probability $p_t^L(\cdot)$ defined by

$$p_t^L(x;y) = \sum_{z\in\mathbb{Z}^d} p_t(x;y+Lz)$$

for $x;y\in\mathbb{T}_L^d$. Since, by definition,

$$E\left[e^{\phi(y+X_t)}\right] = \sum_{x\in\mathbb{T}_L^d} p_t^L(x-y)\,e^{\phi(x)};$$

then, by inverting the order of summation and that $\sum_{y\in\mathbb{T}_L^d} p_t^L(x-y) = 1$, we obtain

$$\mathbb{E}_{\mu_\rho^L}\left[e^{\sum_{x\in\mathbb{T}_L^d}\phi(x)\eta_t(x)}\right] = e^{\rho\sum_{x\in\mathbb{T}_L^d}\left(e^{\phi(x)}-1\right)}.$$

□

Remark 2. Since the total number of particles $\sum_{x\in\mathbb{T}_L^d}\eta(x)$ is conserved by the stochastic dynamics it might seem more natural to consider as reference probability measures the extremal invariant measures that are concentrated on the "hyper-planes" of all configurations with a fixed total number of particles. These measures are given by

$$\mu_{\mathbb{T}_L^d;N}(\cdot) := \mu_\rho^L\left(\cdot \,\middle|\, \sum_{x\in\mathbb{T}_L^d}\eta(x) = N\right).$$

We should, also, note that the Poisson distributions are such that their expectation is equal to

$$\sum_{k \geq 0} e^{-\alpha} \frac{\alpha^k}{k!} k = \alpha :$$

The Poisson measures are in this way naturally parametrized by the density of particles. In addition, by the weak law of large numbers,

$$\lim_{L \to \infty} \frac{1}{|\mathbb{T}_L^d|} \sum_{x \in \mathbb{T}_L^d} \eta(x) = \alpha$$

in probability with respect to ν_α^L. Consequently, imagine that the parameter α describes the mean density of particles in a "large" box.

In conclusion, for this section, we established in proposition 6 the existence of an one-parameter family of invariant measures indexed by the density of particles, which is the unique quantity conserved by the time evolution.

3.3 Local Equilibrium

We said before that one of our main goals is to deduce the macroscopic behavior of an Interacting Particles System. In this case, the passage from microscopic to macroscopic will be done by performing a limit in which the distance between the particle sites will go to zero. This is not difficult to formalize and it is a technique used in many areas of Mathematics.

If we imagine the discrete torus $\mathbb{T}_L^d$ as embedded in a continuous torus $\mathbb{T}^d = [0, 1)^d$, namely taking the lattice $\mathbb{T}^d$ with "vertices" at x/L; $x \in \mathbb{T}_L^d$, then the distances between molecules is $1/L$ and tends to zero as $L \to \infty$. In this way, for the inverse mapping, each macroscopic point u in $\mathbb{T}^d$ is associated to a microscopic site $x = [uL]$ in $\mathbb{T}_L^d$. We should note that, here and below, for a d-dimensional real $r = (r_1, r_2, \ldots, r_d)$, $[r]$ denotes the integer part of r: $[r] = ([r_1], [r_2], \ldots, [r_d])$.

Now, we will start building the concept of a local equilibrium. Let $\rho_0 : \mathbb{T}^d \to \mathbb{R}_+$ be a function describing a density profile. Then, we distribute particles according to a Poisson measure with slowly varying parameter on $\mathbb{T}^d$, that is, for each positive L we fix the parameter of the Poisson distribution at site x to be equal to $\rho_0(x/L)$. This is one way to describe a local equilibrium and since this type of measure will appear frequently, we introduce the following terminology.

Definition 10. For each smooth function $\rho_0 : \mathbb{T}^d \to \mathbb{R}_+$, we represent by $\nu_{\rho_0(\cdot)}^L$ the measure on the state space $\mathbb{N}^{\mathbb{T}_L^d}$ having the following two properties. Under $\nu_{\rho_0(\cdot)}^L$ the variables $(\eta(x) : x \in \mathbb{T}_L^d)$ are independent and, for a site

$x \in \mathbb{T}_L^d$, $\eta(x)$ is distributed according to a Poisson distribution of parameter $\rho_0(x/L)$:

$$\nu^L_{\rho_0(\cdot)}\{\eta;\ \eta(x) = k\} = \nu^L_{\rho_0(x/L)}\{\eta;\ \eta(0) = k\}$$

for all x in $\mathbb{T}_L^d$ and k in $\mathbb{N}$. [11]

Therefore, we have associated to each smooth profile $\rho_0 : \mathbb{T}^d \to \mathbb{R}_+$ and each positive integer L a Poisson measure on the torus $\mathbb{T}_L^d$.

Let us take it one step further and define a limit for the Poisson measures with slowly varying parameter. Notice that, as the parameter L increases to infinity, the discrete torus $\mathbb{T}_L^d$ tends to the full lattice $\mathbb{Z}^d$. We can, also, define a Poisson measure on the space of configurations over $\mathbb{Z}^d$. For each $\alpha \geq 0$ we will denote by ν_α the probability on $\mathbb{N}^{\mathbb{T}_L^d}$ that makes the variables $(\eta(x) : x \in \mathbb{Z}^d)$ independent and under which, for every x in $\mathbb{Z}^d$, $\eta(x)$ is distributed according to a Poisson law of parameter α.

Now, with the definition of $\nu^L_{\rho_0(\cdot)}$, and since $\rho_0 : \mathbb{T}^d \to \mathbb{R}_+$ is assumed to be smooth, as $L \to \infty$ and we look very close to a point $u \in \mathbb{T}^d$, around $x = [Lu]$, we observe a Poisson measure of parameter almost constant and equal to $\rho_0(u)$. In fact, since the function $\rho_0(\cdot)$ is smooth, for every positive integer l and for every positive family of parameters $(\theta(x) : |x| \leq l)$,

$$\lim_{L\to\infty} E_{\nu^L_{\rho_0(\cdot)}}\left[e^{\sum_{|x|\leq l}\theta(x)\eta([uL]+x)}\right] = E_{\nu_{\rho_0(u)}}\left[e^{\sum_{|x|\leq l}\theta(x)\eta(x)}\right]. \tag{3.1}$$

In this sense the sequence $\nu^L_{\rho_0(\cdot)}$ describes an example of local equilibrium. We should note here that, for $u = (u_1, u_2, \ldots, u_d)$ in $\mathbb{R}^d$, $\|u\|$ stands for Euclidean norm of u and $|u|$ the max norm:

$$\|u\|^2 = \sum_{1\leq i\leq d} u_i^2, \quad |u| = \max_{1\leq i\leq d} u_i.$$

In the configuration space $\mathbb{N}^{\mathbb{T}_L^d}$ we denote by $(\tau_x : x \in \mathbb{T}_L^d)$ the group of translations, namely for a site x, $\tau_x\eta$ is the configuration that, at site y, has $\eta(x+y)$ particles:

$$(\tau_x\eta)(y) = \eta(x+y), \ y \in \mathbb{T}_L^d.$$

Keep in mind that the action of the translation group extends in a natural way to the space of functions and to the space of probability measures on $\mathbb{N}^{\mathbb{T}_L^d}$. In fact, for a site x and a probability measure μ, $(\tau_x\mu)$ is the measure such that

$$\int f(\eta)(\tau_x\mu)(d\eta) = \int f(\tau_x\eta)\,\mu(d\eta)$$

for all bounded continuous f.

With this topological setting, equation 3.1 expresses that for all points $u \in \mathbb{T}^d$, the sequence $\tau_{[uL]}\nu^L_{\rho_0(\cdot)}$ converges weakly to the measure $\nu^L_{\rho_0(u)}$. Finally, we will present the following definition.

Definition 11. A sequence of probability measures $(\mu^L)_{L \geq 1}$ on $\mathbb{N}^{\mathbb{T}^d_L}$ is a local equilibrium of profile $\rho_0 : \mathbb{T}^d \to \mathbb{R}_+$ if

$$\lim_{L \to \infty} \theta_{[uL]} \mu^L = \nu_{\rho_0(u)}$$

for all continuity points u of $\rho_0(\cdot)$. [11]

3.4 Macroscopic Profile

3.4.1 Scaling Limits

In this section, we will address the matter of what will be the macroscopic profile of our system after time t. We assume that the initial state will follow a product measure with slowly varying parameter as defined in definition 10. We see that if we start from a Poisson measure with slowly varying parameter then

$$\begin{aligned} E_{\nu^L_{\rho_0(\cdot)}}\left[e^{-\sum_{x \in \mathbb{T}^d_L} \lambda(x)\eta_t(x)} \right] &= \\ & e^{\sum_{x \in \mathbb{T}^d_L} \rho_0(x/L) \sum_{y \in \mathbb{T}^d_L} p^L_t(y-x)\left(e^{-\lambda(y)}-1\right)} = \\ & e^{\sum_{y \in \mathbb{T}^d_L} \left(e^{-\lambda(y)}-1\right) \sum_{x \in \mathbb{T}^d_L} p^L_t(y-x)\rho_0(x/L)} =: \\ & e^{\sum_{y \in \mathbb{T}^d_L} \left(e^{-\lambda(y)}-1\right) \rho_{L,t}(y)} \end{aligned}$$

In the above equation [11] the first step is reached by repeating the same computations we did to prove proposition 6, and then by inverting the order of summation, we get that, at time t, we still have a Poisson measure with slowly varying parameter, which is now $\rho_{L,t}(\cdot)$ instead of $\rho_0(\cdot/L)$.

It is true that, up to this point, we have not really discussed much about $p_t(\cdot)$ and how it affects the system. We have only said that it makes $p_t(\cdot,\cdot)$ translation invariant and thus bistochastic: $\sum_x p_t(x,y) = 1$ for every y. Let us now see what happens when t is fixed and L increases to infinity. In this case $p_t(\cdot)$ is a function with essentially finite support, that is, for all $\varepsilon > 0$, there exists $A = A(t,\varepsilon) > 0$ so that

$$\sum_{|x| \leq |A|} p_t(x) \geq 1 - \varepsilon.$$

From the explicit form of $\rho_{L,t}$, we have that for every continuity point u of ρ_0,

$$\lim_{L \to \infty} \rho_{L,t}([uL]) = \rho_0(u).$$

What the above equation tells us is that the profile remained unchanged. Even though time t have passed, it seems that the system did not have enough time to evolve and this reflects the fact that at the macroscopic scale the particles did not move. Consider the following test: select a particle at the origin; since it evolves as a discrete time random walk, if X_t denotes its position at time t, for every $\varepsilon > 0$, there exists $A = A(t; \varepsilon) > 0$ such that $P(|X_t| > A) \leq \varepsilon$. Therefore, with probability close to 1, in the macroscopic scale, the test particle at time t is at distance of order L^{-1} from the origin.

In order to solve this problem, we need to distinguish between two different time scales, as we already have different space scales, $\mathbb{T}^d$ and $L^{-1}\mathbb{T}^d_L$. Respectively, we need a microscopic time t and a macroscopic time which would be infinitely large with respect to t.

To introduce the macroscopic time scale, notice that the transition probabilities $p_t(\cdot)$ are equal to

$$p_t(x) = \sum_{k=0}^{\infty} e^{-t} \frac{t^k}{k!} p^{(k)}(x);$$

where $p^{(k)}$ stands for the k-th convolution power of the elementary transition probability of each particle.

Assume that the elementary transition probability $p(\cdot)$ has finite expectation: $m := \sum x p(x) \in \mathbb{R}^d$. We say that the random walk is *asymmetric* if $m \neq 0$, that it is *mean-zero asymmetric* if $p(\cdot)$ is not symmetric but $m = 0$ and that it is *symmetric* if $p(\cdot)$ is symmetric. Recall that X_t stands for the position at time t of a discrete time random walk with transition probability $p(\cdot)$ and initially at the origin. By the law of large numbers for random walks, for all $\varepsilon > 0$,

$$\lim_{L \to \infty} \sum_{x; |x/L - mt| \leq \varepsilon} p_{tL}(x) = \lim_{L \to \infty} P\left(\left| \frac{X_{tL}}{L} - mt \right| \leq \varepsilon \right) = 1$$

In particular, from the explicit expression for $\rho_{L;tL}$ and since we assumed the initial profile to be smooth, we have that

$$\lim_{L \to \infty} \rho_{L;tL}([uL]) = \rho_0(u - mt) =: \rho(t; u)$$

for every u in $\mathbb{T}^d$.

Now you can see that with the new time scale, tL the profile did not remain unchanged. On the contrary, we observe a new macroscopic profile: the original one translated by mt. More precisely, in this macroscopic scale tL we observe a local equilibrium profile that has been translated by mt since $\rho_{L;tL}$ is itself slowly varying in the macroscopic scale.

3.4.2 Hydrodynamic Equation

Of course, the profile $\rho(t; u)$ satisfies the partial differential equation

$$\partial_t \rho + m \cdot \nabla \rho = 0$$

if $\nabla \rho$ denotes the gradient of ρ: $\nabla \rho = (\partial_{u_1} \rho, \partial_{u_2} \rho, \ldots, \partial_{u_d} \rho)$.

We see that if we restrict ourselves to a particular class of initial measures, we are able to establish the existence of a time and space scales in which the particles density evolves according to the linear partial differential equation. Consequently, we have derived from the microscopic stochastic dynamics a macroscopic deterministic evolution for the unique conserved quantity.

An interacting particle system for which there exists a time and space macroscopic scales in which the conserved quantities evolve according to some partial differential equation is said to have a *hydrodynamic description*. Moreover, the partial differential equation is called the *hydrodynamic equation* associated to the system.

Proposition 7. *A system of particles evolving as independent asymmetric random walks with finite first moment on a d-dimensional torus has a hydrodynamic description. The evolution of the density profile is described by the solution of the differential equation*

$$\partial_t \rho + m \cdot \nabla \rho = 0.$$

However, when the random walk is not asymmetric and the expectation m vanishes, the solution of this differential equation is constant, which means that the profile didn't change in the time scale tL. This makes sense intuitively, as the system now is unbiased in direction. Still, if we consider a larger time scale, times of order L^2, even when the mean displacement m vanishes, we can observe an interesting time evolution. [11]

Let $(S^L(t) : t \geq 0)$ be the semigroup associated to the Markov process $(\eta_t)_{t \geq 0}$. In Proposition 7, we have proved that there is a time renormalization θ_L such that

$$\lim_{L \to \infty} S^L(t\theta_L)\tau_{[uL]}\nu^L_{\rho_0(\cdot)} = \nu_{\rho(t,u)},$$

for all $t \geq 0$ and all continuity points u of $\rho(t,\cdot)$.

All in all, we have proved in this chapter the following results:

- Description of the equilibrium states of the system.
- Conservation of the local equilibrium in time evolution.
- Characterization at a later time of the new parameters describing the local equilibrium and derivation of a partial differential equation that determines how the parameters evolve in time.

3.5 Equivalence of Ensembles

It is true that, we have chose a class of invariant measures to describe the equilibrium states (the Poisson measures) when others would seem more appropriate. Following remark 2 and the fact that we want to describe the equilibrium state associated to a given density on the torus $\mathbb{T}^d$, we would be led to study the behavior, as $L \to \infty$ and α is kept fixed, of

$$\nu_\alpha^L \left(\cdot \;\Big|\; \sum_{x \in \mathbb{T}_L^d} \eta(x) = L^d \alpha \right) .$$

Performing a simple computation presented below, we get that for each fixed positive integer r, and for all sequences $(k_1, \dots, k_r)$ in $\mathbb{N}^r$ and $(x_1, \dots, x_r)$ in $\mathbb{Z}^r$,

$$\lim_{L \to \infty} \nu_\alpha^L \left(\eta(x_1) = k_1, \dots, \eta(x_r) = k_r \;\Big|\; \sum_{x \in \mathbb{T}_L^d} \eta(x) = L^d \alpha \right)$$

$$= \nu_\alpha(\eta(x_1) = k_1, \dots, \eta(x_r) = k_r) .$$

Indeed, it is easy to check by computing the Laplacian transform that the addition of independent Poisson distributions, is still a Poisson distribution with parameter equal to the sum of the parameters. Consequently, the left hand side of the above formula is equal to

$$\frac{\alpha^{k_1+\cdots+k_r}}{k_1! \cdots k_r!} e^{-\alpha L^d} \frac{((L^d - r)\alpha)^{L^d\alpha - (k_1+\cdots+k_r)}}{(L^d\alpha - (k_1 + \cdots + k_r))!} \left(e^{-\alpha L^d} \frac{(L^d\alpha)^{L^d\alpha}}{(L^d\alpha)!} \right)^{-1} =$$

$$\frac{(L^d - r)^{L^d\alpha - (k_1+\cdots+k_r)}}{k_1! \cdots k_r! (L^d)^{L^d\alpha}} (L^d\alpha)(L^d\alpha - 1) \cdots (L^d\alpha - (k_1 + \cdots + k_r) + 1)$$

which, as $L \to \infty$, converges to

$$\frac{\alpha^{k_1+\cdots+k_r}}{k_1! \cdots k_r!} e^{-\alpha r} .$$

So, we see that the Poisson measures are "natural" for our system and at the same time computations are made much easier and the definition of local equilibrium is expressed in a very simple and elegant way in terms of these measures. In this sense, this fact is known to the physicists as the "equivalence of ensembles".

Chapter 4

Simple Exclusion Process

In this chapter we are interested in Simple Exclusion processes (SEP). This model was introduced in [16] and it is among the simplest and most widely studied interacting particle systems. The Simple Exclusion process, in contrast with the independent random walks studied in the previous chapter, allows at most one particle per site.

4.1 Model

First of all, the state space is $\{0,1\}^{\mathbb{T}_L^d}$. In order to prevent the occurrence of more than one particle per site we introduce an exclusion rule that suppresses each jump to an already occupied site. In fact, we shall focus only on the simplest class of exclusion processes: systems where particles jump, whenever the jump is allowed, independently of the others and according to the same translation invariant elementary transition probability.

Definition 12 (Elementary jump probability). Let p be a finite range, translation invariant, irreducible transition probability on $\mathbb{Z}^d$:

$$p(x,y) = p(0,y-x) =: p(y-x)$$

for all pair (x,y) of d-dimensional integers and for some finite range probability measure $p(\cdot)$ on $\mathbb{Z}^d$:

$$\sum_{z\in\mathbb{Z}^d} p(z) = 1 \text{ and } p(x) = 0 \text{ for } |x| \text{ large enough. [11]}$$

The generator

$$(Lf)(\eta) := \sum_{x\in\mathbb{T}_L^d} \sum_{z\in\mathbb{T}_L^d} \eta(x)(1-\eta(x+z))p^L(z)(f(\eta^{x,x+z}) - f(\eta));$$

where $\eta^{x,y}$ is the configuration obtained from η letting a particle jump from x to y, namely

$$\eta^{x,y}(z) = \begin{cases} \eta(z) & \text{if } z \neq x,y, \\ \eta(x) - 1 & \text{if } z = x, \\ \eta(y) + 1 & \text{if } z = y \end{cases} \quad \text{and} \quad p^L(z) := \sum_{y \in \mathbb{Z}^d} p(z + yL);$$

defines a Markov process called simple exclusion process with elementary jump probability $p(\cdot)$. In the particular case where $p(z) = p(-z)$ we say that it is a symmetric simple exclusion process.

We believe that the interpretation is clear. Between 0 and dt each particle tries, independently from the others, to jump from x to $x+z$ with rate $p^L(z)$. The jump is suppressed if it leads to an already occupied site.

We remind here that a Markov process is said to be irreducible if it is possible to get to any state from any state. Furthermore, since the transition probability is assumed to be of finite range, there exists A_0 in $\mathbb{N}$ such that $p(z) = 0$ for all sites outside the cube $[-A_0, A_0]^d$. In particular, $p^L(\cdot)$ and $p(\cdot)$ coincide provided $L \geq A_0$. For this reason, from now on we omit the superscript L in the elementary jump probability.

At this point, it might be worthwhile to justify the terminology. The rule that forbids jumps to occupied sites explains the term exclusion. Notice, on the other hand, that the rate at which a particle jumps from x to y depends on the configuration η only through the occupation variables $\eta(x)$ and $\eta(y)$. This last dependence on $\eta(x)$ and $\eta(y)$ reflects the exclusion rule. Finally, notice that the total number of particles is conserved by the dynamics.

4.2 Bernoulli Measure

We denote by $\nu_\alpha = \nu_\alpha^L$, for $0 \leq \alpha \leq 1$, the Bernoulli product measure of parameter α, that is, the product and translation invariant measure on $\{0,1\}^{\mathbb{T}_L^d}$ with density α. In particular, under ν_α, the variables $(\eta(x) : x \in \mathbb{T}_L^d)$ are independent with marginals given by

$$\nu_\alpha\{\eta(x) = 1\} = \alpha = 1 - \nu_\alpha\{\eta(x) = 0\}.$$

Proposition 8. *The Bernoulli measures $\{\nu_\alpha : 0 \leq \alpha \leq 1\}$ are invariant for simple exclusion processes. In addition, with respect to each ν_α, exclusion processes with elementary jump probability $p^*(z) := p(-z)$ are adjoint to processes with elementary jump probability $p(z)$. In particular, symmetric simple exclusion processes are self-adjoint with respect to each ν_α. [11]*

Proof. It is easy to notice that by a simple change of variables

$$\int f(\eta^{0,z}) g(\eta)\, \eta(0)(1 - \eta(z))\, \nu_\alpha(d\eta) = \int f(\eta) g(\eta^{0,z})\, \eta(z)(1 - \eta(0))\, \nu_\alpha(d\eta).$$

This identity, the fact that $1 = \sum_{z \in \mathbb{Z}^d} p(z) = \sum_{z \in \mathbb{Z}^d} p(-z)$ and a change in the order of summation prove the proposition. □

In this case, the family of invariant measures ν_α is parametrized by the density, for

$$E_{\nu_\alpha}[\eta(0)] = \nu_\alpha\{\eta : \eta(0) = 1\} = \alpha.$$

Remark 3. Since the total number of particles is conserved by the dynamics, the measures

$$\nu_{L,N}(\cdot) := \nu\left(\cdot \;\middle|\; \sum_{x\in\mathbb{T}_L^d} \eta(x) = N\right).$$

are invariant and it could have seemed more natural to consider them instead of the Bernoulli product measures ν_α. Nevertheless, a simple computation on binomials shows that for all finite subsets E of $\mathbb{Z}^d$, for all sequences $\{\alpha_x : x \in E\}$ with values in $\{0,1\}$ and for all $0 \le \alpha_0 \le 1$,

$$\lim_{L\to\infty} \nu\left\{\eta(x) = \alpha_x;\, x \in E \;\middle|\; \sum_{y\in\mathbb{T}_L^d} \eta(y) = [\alpha_0 L^d]\right\} =$$

$$\nu_{\alpha_0}\{\eta : \eta(x) = \alpha_x;\, x \in E\}$$

uniformly in α_0. Consequently, the Bernoulli product measures are obtained as limits of the invariant measures $\nu_{L,N}$, as the total number of sites increases to infinity.

4.3 Asymmetric Simple Exclusion Process with step initial condition

One extensively studied variance of the simple exclusion process is the one-dimensional nearest neighbour asymmetric simple exclusion process with step initial condition. In this area, many significant results have been achieved by Tracy and Widom.

Let us consider the integer lattice $\mathbb{Z}$. In the case of step initial condition, particles will begin from the positive integers $\mathbb{Z}_+$. As you know, a particle waits exponential time, then moves to the right with probability p if that site is unoccupied or to the left with probability $q = 1 - p$ if that site is unoccupied. If the site where it is about to jump is occupied, then it stays put.

The main quantity that will concern us in this section is the position of the mth particle from the left at time t, denoted by

$$x_m(t), \text{ with } x_m(0) = m.$$

Here we shall, also, assume that $p < q$, so there is a drift to the left, and establish results on the position of the mth particle and the current of particles. Now we will start by describing the results which were presented and proved in [21].

The authors in [17], derived a formula for the quantity that interests us, valid when p and q are non-zero. It is given in terms of the Fredholm determinant of a kernel $K(\xi;\xi')$ on C_R, a circle with center zero and large radius R described counterclockwise. The Fredholm determinant of a kernel K is the operator determinant $\det(I-K)$. It acts as an operator by

$$f(\xi) \to \int_{C_R} K(\xi;\xi') f(\xi') d\xi'$$

for all $\xi \in C_R$. We will use the following notation

$$\gamma = q - p; \qquad \tau = q/p.$$

The kernel is

$$K(\xi;\xi') = q \frac{\xi'^x e^{\varepsilon(\xi')t}}{p + q\xi\xi' - \xi};$$

where

$$\varepsilon(\xi) = p\xi^{-1} + q\xi - 1.$$

The first formula is the following

$$\mathbb{P}(x_m(t/\gamma) \le x) = \int \frac{\det(I-\lambda K)}{\prod_{k=0}^{m-1}(1-\lambda\tau^k)} \frac{d\lambda}{\lambda}. \tag{4.1}$$

The integral is taken over a contour enclosing the singularities of the integrand at $\lambda = 0$ and $\lambda = \tau^{-k}$, $k = 0, \ldots, m-1$. It is easily derived from the above equation that

$$\mathbb{P}(x_1(t/\gamma) > x) = \det(I - K).$$

It is clear probabilistically that $\mathbb{P}(x_m(t) > x) = 0$ for all t when $x \ge m$, as for a particle to be to the right of its initial position, all particles would have to move simultaneously to the right, which surely has probability zero.

Although, the above formula required $p > 0$, the statement makes sense when $p = 0$. The process where $p = 0$ and the particles move only to the left is called Totally asymmetric simple exclusion process.

For the first asymptotic result, denote by $\hat{K}$ the operator on $L^2(\mathbb{R})$ with kernel,

$$\hat{K}(z;z') = \frac{q}{\sqrt{2\pi}} e^{-(p^2+q^2)(z^2+z'^2)/4 + pqzz'}.$$

Assume that $0 < p < q$. For fixed m the limit

$$\lim_{t\to\infty} \mathbb{P}\left(\frac{x_m(t/\gamma) + t}{2^{1/2} t^{1/2}} \le s\right)$$

is equal to the integral 4.1 with K replaced by the operator $\hat{K}\chi_{(-s,\infty)}$. It is easy again to derive the special case

$$\lim_{t\to\infty} \mathbb{P}\left(\frac{x_1(t/\gamma) + t}{2^{1/2} t^{1/2}} > -s\right) = \det(1 - \hat{K}\chi_{(s,\infty)}).$$

This is an apparently new family of distribution functions, parametrized by p. When $p = 0$ the kernel has rank one and the determinant equals a standard normal distribution.

Furthermore, we will state the asymptotic result when m and x both go to infinity. We use the notation

$$\sigma = m/t; \quad c_1 = -1 + 2\sqrt{\sigma}; \quad c_2 = \sigma^{-1/6}(1 - \sqrt{\sigma})^{2/3}.$$

Theorem 2. *When* $0 \le p < q$ *we have*

$$\lim_{t \to \infty} \mathbb{P}\left(\frac{x_m(t/\gamma) - c_1 t}{c_2 t^{1/3}} \le s \right) = F_2(s)$$

uniformly for σ *in a compact subset of* $(0, 1)$.

In the above theorem, the function $F_2(s)$ that arises asymptotically is the Tracy-Widom distribution. The Tracy-Widom distributions are a family of probability distributions that were described explicitly by Craig Tracy and Harold Widom [18, 19], and shown to govern the maximal eigenvalue of large random matrices.

The cumulative distribution function of the Tracy-Widom distribution can be given as an integral

$$F_2(s) = e^{-\int_s^\infty (x - s) q^2(x) dx} \tag{4.2}$$

and q is the unique solution to the Painlevé II equation

$$q'' = sq + 2q^3; \text{ with boundary condition } q(s) \sim Ai(s) \text{ as } s \to \infty;$$

where Ai is the Airy function.

Let us now introduce the following quantity. As the particles are initially located at $\mathbb{Z}_+$ and we have assumed that $p < q$, then there will be on average a net flow of particles, or current, to the left. The *total current* I at position $x \le 0$ at time t,

$$I(x; t) := \text{number of particles with position} \le x \text{ at time } t.$$

With step initial condition, it stands that, for $0 \le c \le \gamma$, the current I satisfies the strong law [12],

$$\lim_{t \to \infty} \frac{I([-ct]; t)}{t} = \frac{1}{4}(\gamma - c)^2.$$

Now, we want to examine the behavior of the current fluctuations

$$I(x; t) - \frac{1}{4}(\gamma - c)^2 t$$

for large x and t. It has been proved [10, 22] that to obtain a nontrivial limiting distribution the correct normalization of the fluctuations is cube root in t.

Theorem 3. *For an asymmetric simple exclusion process with step initial condition we have, for* $0 \le \sigma < 1$,

$$\lim_{t\to\infty} \mathbb{P}\left(\frac{l([\xi t]; t/\gamma) - \alpha_1 t}{\alpha_2 t^{1/3}} \le s\right) = 1 - F_2(-s),$$

where $\alpha_1 = \frac{1}{4}(1-\sqrt{\xi})^2$ *and* $\alpha_2 = 2^{-4/3}(1-\xi^2)^{2/3}$. *[22]*

Proof. We are interested in the probability of the event,

$$\{l(x;t) = m\} = \{x_m(t) \le x, x_{m+1}(t) > x\}.$$

The sample space consists of the four disjoint events $\{x_m(t) \le x, x_{m+1}(t) > x\}$, $\{x_m(t) \le x, x_{m+1}(t) \le x\}$, $\{x_m(t) > x, x_{m+1}(t) > x\}$, and $\{x_m(t) > x, x_{m+1}(t) \le x\}$, and because of the exclusion property, we have

$$\{x_m(t) \le x, x_{m+1}(t) \le x\} = \{x_{m+1}(t) \le x\},$$

$$\{x_m(t) > x, x_{m+1}(t) > x\} = \{x_m(t) > x\},$$

$$\{x_m(t) > x, x_{m+1}(t) \le x\} = \emptyset.$$

These observations result to the intuitively obvious

$$\mathbb{P}(l(x;t) = m) = \mathbb{P}(x_m(t) \le x) - \mathbb{P}(x_{m+1}(t) \le x).$$

And, since $\mathbb{P}(l(x;t) = 0) = \mathbb{P}(x_1(t) > x)$, we have

$$\mathbb{P}(l(x;t) \le m) = 1 - \mathbb{P}(x_{m+1}(t) \le x).$$

Therefore, since x and $x_{m+1}(t)$ are integers, the statement of the Theorem is equivalent to the statement that

$$\lim_{t\to\infty} \mathbb{P}(x_{m+1}(t/\gamma) \le \xi t) = F_2(s),$$

when $m = [\alpha_1 t - \alpha_2 s t^{1/3}]$. In particular, we shall show that

$$\lim_{t\to\infty} \mathbb{P}(x_m(t/\gamma) \le \xi t) = F_2(s),$$

when

$$m = \alpha_1 t - \alpha_2 s t^{1/3} + o(t^{1/3}).$$

Now, in order to obtain the last limit from Theorem 2, we determine σ so that

$$\xi t = c_1 t + c_2 s t^{1/3}.$$

Thus, after some computations, we get that

$$\sigma = \left(\frac{1-\sqrt{\xi}}{2}\right)^2 - s2^{-4/3}(1-\xi^2)^{2/3}t^{-2/3} + o(t^{-2/3}).$$

Since this is exactly the statement that $m = \sigma t$ must satisfy, we see that the Theorem is established. □

4.4 Universality of the Tracy-Widom Distribution

In this section, we will discuss about the universality that Tracy-Widom distribution has been found to exhibit lately. We will start by understanding how we ended up in such hypothesis.

It is a fact that Random Matrix Theory has found a huge number of applications ranging from statistical physics of disordered systems, quantum information, finance, telecommunication networks to number theory, combinatorics and integrable systems. Among the recent developments in Random Matrix Theory, the study of the largest eigenvalue λ_{max} of large random matrices has attracted particular attention. The first questions were related to the fluctuations of λ_{max}, belonging to the wider topic of extreme value statistics. Such extreme value questions arise naturally in the statistical physics of complex and disordered systems like interacting particle systems. In particular, the eigenvalues of a random matrix provide an interesting "toolset" to study extreme value statistics of strongly correlated random variables.

Biologist Robert May realized in 1972 a natural application of the statistics of λ_{max}, which is to provide a criterion of physical stability in dynamical systems such as ecosystems [14]. May considered a population of N distinct species and introduced strong pair-wise interactions between the species. May assumed that the interactions between pairs of species can be modeled by a random matrix J, of size $N \times N$, which is real and symmetric. A natural question is then: what is the probability, $P_{stable}(\alpha; N)$, where α represents the strength of interactions, that the system remains stable once the interactions are switched on? [14] After some computations, May derived that the system will remain stable, provided the eigenvalues λ_i of the random matrix J satisfy the inequality:

$$\alpha\lambda_i - 1 < 0;$$

for all $i = 1; \ldots; N$. This is obviously equivalent to the statement that the largest eigenvalue $\lambda_{max} = \max_{1 \le i \le N} \lambda_i$ satisfies the inequality:

$$\lambda_{max} < \frac{1}{\alpha}:$$

Hence the probability that the system is stable gets naturally related to the cumulative distribution function of the largest eigenvalue λ_{max}.

The Tracy-Widom distribution was first established in 1992, two decades later, by Tracy and Widom [18, 19], who observed it by studying the same concept, namely the fluctuations of the largest eigenvalue λ_{max} of random matrices. Later in 1999, Baik, Deift and Johansson [2], discovered that the same statistical distribution also describes variations in sequences of shuffled integers - a completely unrelated mathematical abstraction. Specifically, let S_N be the group of all permutations of N numbers with uniform distribution and let $l_N(\sigma)$ be the length of the longest increasing subsequence of $\sigma \in S_N$.

Let be a random variable whose distribution function is F_2 (4.2). Then, as $N \to \infty$

$$\frac{l_N - 2\sqrt{N}}{N^{1/6}} \to$$

in distribution.

Soon the distribution started to appear in models all over physics and mathematics. Systems of many interacting components kept producing the same statistical curve. This puzzling curve seemed to be the complex cousin of the familiar bell curve, or Gaussian distribution, which represents the natural variation of independent random variables. Like the Gaussian, the Tracy-Widom distribution exhibits universality, a mysterious phenomenon in which diverse microscopic effects give rise to the same collective behavior.

The Tracy-Widom distribution is an asymmetrical statistical bump which is steeper on the left side than the right. Suitably scaled, its summit sits at a telltale value: $\sqrt{2N}$, the square root of twice the number of variables in the systems that give rise to it.

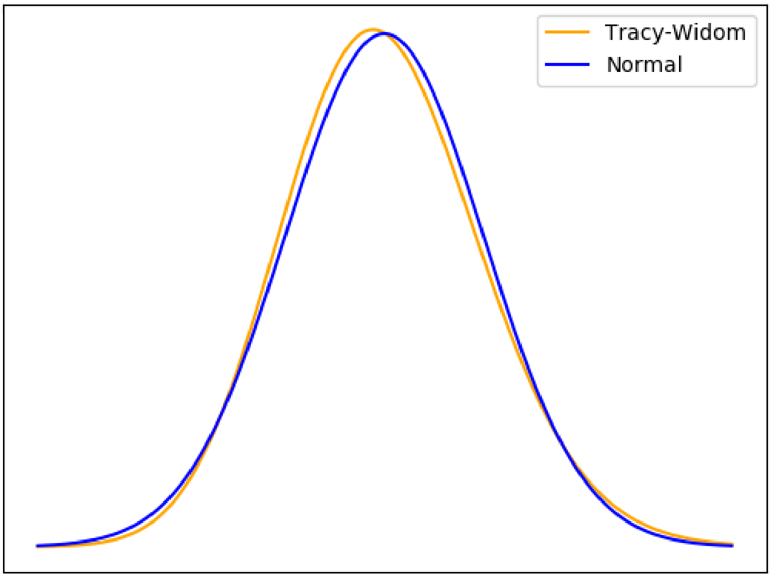

Figure 4.1: Probability density functions of Normal and Tracy-Widom distributions

When the Tracy-Widom distribution turned up in the integer sequences problem and other contexts that had nothing to do with random matrix theory, researchers began searching for the "hidden thread" tying all its manifestations together, just as mathematicians in the 18th and 19th centuries sought a theorem that would explain the ubiquity of the Normal distribution.

The Central Limit Theorem, which was finally made rigorous about a century ago, certifies that natural observations and other "uncorrelated" variables - meaning any of them can change without affecting the rest - will form a bell curve. By contrast, the Tracy-Widom curve appears to arise from variables that are strongly correlated, such as interacting species, stock prices and matrix eigenvalues. The feedback loop of mutual effects between correlated variables makes their collective behavior more complicated than that of uncorrelated variables.

While researchers have rigorously proved certain classes of random matrices in which the Tracy-Widom distribution universally holds, they have a looser handle on its manifestations in counting problems, random walk problems, growth models and beyond.

So far, there have been characterized three forms of the Tracy-Widom distribution: rescaled versions of one another that describe strongly correlated systems with different types of inherent randomness, namely different types of random matrix ensembles. Specifically, the three classes of $F(s)$ are indexed by

$$= \begin{cases} 1, & \text{for Gaussian orthogonal ensemble} \\ 2, & \text{for Gaussian unitary ensemble} \\ 4, & \text{for Gaussian symplectic ensemble} \end{cases} .$$

But there could be many more than three, perhaps even an infinite number, of Tracy-Widom universality classes.

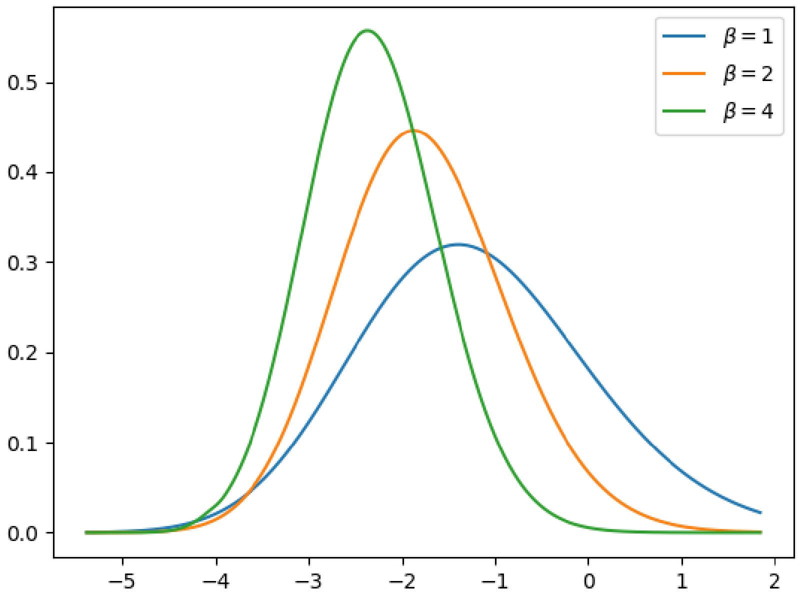

Figure 4.2: Classes of Tracy-Widom distribution

Lately it is being extensively discussed the fact that the asymmetric character of the distribution may represent some kind of universal phase transition [13]. In May's ecosystem model, for example, the critical point at $\sqrt{2N}$ separates a stable phase of weakly coupled species, whose populations can fluctuate individually without affecting the rest, from an unstable phase of strongly coupled species, in which fluctuations cascade through the ecosystem and throw it off balance. In general, systems in the Tracy-Widom universality class exhibit one phase in which all components act in concert, left tail, and another phase in which the components act alone, right tail.

Right now, many physicists and mathematicians are working in the field of seeking some universal law tied to the Tracy-Widom distribution. If such breakthrough were to be achieved, we would be able to interpret the macroscopic elements of systems with interacting components in a much more natural way.

Chapter 5

Zero Range Process

In this chapter we will study another widely known model of Interacting Particle Systems. It was also originally introduced as a simple example of an interacting Markov process in [16]. It is called Zero Range Process and its name originates from the fact that the particles will only interact with particles sitting on the same site.

5.1 Model

As in chapter 3, we will consider evaluations without restrictions on the total number of particles per site. The state space will therefore be $\mathbb{N}^{\mathbb{T}_L^d}$. The process is defined through a function $g : \mathbb{N} \to \mathbb{R}_+$ vanishing at zero, which represents the rate at which one particle leaves a site, and a translation invariant transition probability $p(\cdot;\cdot)$ on $\mathbb{Z}^d$. Its dynamics goes as follows. If there are k particles at a site x, independently of the number of particles on other sites, at rate $g(k)p(x;y)$ one of the particles at x jumps to y. In this way particles interact only with particles in the same site.

Definition 13. Let $g : \mathbb{N} \to \mathbb{R}_+$ be a function with $g(0) = 0$ and $p(\cdot;\cdot)$ be a finite range, irreducible, translation invariant transition probability. We assume that g is strictly positive on the set of positive integers and that it has bounded variation in the following sense:

$$g^* := \sup_{k \geq 0} |g(k+1) - g(k)| < \infty.$$

Now, let $Z : \mathbb{R}_+ \to \mathbb{R}_+$ be the partition function defined by

$$Z(\varphi) = \sum_{k \geq 0} \frac{\varphi^k}{g!(k)}$$

and denote by φ_c the radius of convergence of Z. In the last formula $g!(k)$ stands for $\prod_{1 \leq j \leq k} g(j)$ and by convention $g!(0) = 1$. Furthermore, notice that Z is analytic and strictly increasing in $[0, \phi_c)$. Assume that $Z(\phi)$ increases to ∞ as ϕ converges to ϕ_c:

$$\lim_{\phi \to \phi_c} Z(\phi) = \infty. \tag{5.1}$$

This assumption is not necessary to define the process, but will always be required to prove following results. The generator

$$(Lf)(\eta) = \sum_{x \in \mathbb{T}_L^d} \sum_{z \in \mathbb{T}_L^d} p^L(z) g(\eta(x)) (f(\eta^{x,x+z}) - f(\eta))$$

defines a Markov process on $\mathbb{N}^{\mathbb{T}_L^d}$, called zero range process with parameters (g, p). Also here, as in chapter 4, $\eta^{x,y}$ represents the configuration where one particle jumped from x to y and $p^L(\cdot)$ represents the transition probability translated to the origin and restricted to the torus:

$$p^L(z) := p^L(0, z) = \sum_{y \in \mathbb{T}_L^d} p(0, z + yL)$$

for every d-dimensional integer z.

In zero range processes each particle jumps, independently of particles sitting at other sites, from x to y at rate

$$p^L(y - x) g(\eta(x)) \frac{1}{\eta(x)}.$$

In particular, if $g(k) = k$ for every $k \geq 0$, we obtain the superposition of independent random walks studied in chapter 3. On the other hand, the case $g(k) = 1\{k \geq 1\}$ models a system of queues with mean-one exponential random times of service. Moreover, we will study, later, the case where g is a decreasing function, and also perform experiments on it.

5.2 Invariant measures

We now turn our attention, as usual, to the characterization of invariant measures for the process. Since the zero range process is irreducible and the state space is finite, we have a unique invariant measure which we denote by $\nu_{N,L}$. We will refer to the measures $\nu_{N,L}$ as the canonical ensembles. They can be explicitly computed, but they can also be obtained by conditioning the grand-canonical ensembles, whose definition follows, on the total number of particles.

Definition 14. For each $0 \leq \phi < \phi_c$, let $\bar{\nu}_{\phi,g} = \nu^L_{\phi,g}$ denote the product measure on $\mathbb{N}^{\mathbb{T}_L^d}$ with marginals given by

$$\bar{\nu}_{\phi,g}\{\eta;\ \eta(x) = k\} = \frac{1}{Z(\phi)} \frac{\phi^k}{g!(k)}$$

for each $k \geq 0$ and $x \in \mathbb{T}_L^d$.

Proposition 9. *For each $0 \le \phi < \phi_c$ the product measure $\nu_{\phi;g}$ is invariant for the zero range process with parameters $(g;p)$. Moreover, the adjoint process with respect to any of the measures $\nu_{\phi;g}$ is the zero range process with parameters $(g;p^*)$. In particular, if p is symmetric the process is self-adjoint. [11]*

Proof. The proof relies on the same computations we did for Proposition 8 and on the following identity

$$g(k)\frac{\phi^k}{g!(k)}\frac{\phi^j}{g!(j)} = g(j+1)\frac{\phi^{k-1}}{g!(k-1)}\frac{\phi^{j+1}}{g!(j+1)}.$$

□

Also, because the function $g(\cdot)$ will always be fixed, to keep notation simple, we omit the dependence on g of the measure $\nu_{\phi;g}$ and denote it simply by ν_ϕ. And so,

$$\mu_{N;L}(\cdot) = \nu_\phi\left(\cdot \;\middle|\; \sum_{x\in \mathbb{T}_L^d} \eta(x) = N\right).$$

Now, let $R(\phi)$ denote the expected value of the occupation variable, namely the density, under ν_ϕ:

$$R(\phi) = E_{\nu_\phi}[\eta(x)] = \frac{1}{Z(\phi)}\sum_{k\ge 0} k\frac{\phi^k}{g!(k)}. \tag{5.2}$$

The range of R is the interval $[0;\rho_c)$, with $R(0) = 0$ and

$$\rho_c = \lim_{\phi\to\phi_c} R(\phi)$$

the critical density. Also, equation 5.2 can easily be transformed into the following relation, which is usually seen in the concept of partition functions and will be often used later:

$$R(\phi) = \phi\frac{Z'(\phi)}{Z(\phi)} = \phi\,\partial_\phi \log Z(\phi). \tag{5.3}$$

Computing the first derivative of $R(\cdot)$ shows that it is strictly increasing.

Remark 4. A natural object of interest is to explore the behavior of these measures in the thermodynamic limit, namely as $N;L \to \infty$ in such a way that the average particle density N/L converges to a constant ρ. Well, in the subcritical case, when $\rho < \rho_c$, there exists a fugacity ϕ such that $\rho = R(\phi)$ and the standard equivalence of ensembles for independent random variables holds [11]. That is, the finite dimensional marginals of the canonical ensembles $^{N;L}$ converge to the grand-canonical ensemble corresponding to fugacity . The equivalence of ensembles for critical and supercritical densities, when $_c$, was established in [7]. Using relative entropy methods the authors prove convergence of the finite dimensional marginals of $^{N;L}$ to the grand-canonical ensemble at critical fugacity. Later in [1], the authors showed that in the thermodynamic limit the sites have joint distribution equal to the grand canonical measure at critical density, expect one site which accommodates a macroscopically large number of particles.

Furthermore, we have often parametrized the invariant measures by the conserved quantity, which here is the density of particles. For that reason, we change variables in the definition of the invariant measures as follows. For 0, define the product measure by

$$( ) = \quad _{( )}( );$$

where ( ) stands for

$$( ) = \begin{cases} \text{inverse of } ( ), & \text{for } < _c \\ _c, & \text{for } _c \end{cases}$$

In the next lemma we show that assumption 5.1 guarantees that the range of the function ( ) is all $\mathbb{R}_+$. In this way, we obtained a family f : 0g of invariant measures parametrized by the density, since the expected value of the occupation variables (x) under is equal to :

$$E \; [(x)] = E_{( )}[(x)] = (( )) =$$

for every 0. Moreover, a simple computation shows that the function ( ) is the expected value of the jump rate $g((0))$ under the measure :

$$( ) = E \; [g((0))]:$$

Lemma 1. *Recall that we denoted by $_c$ the radius of convergence of the partition function Z.*

$$_c = \lim_{! \; c} ( ) = 1:$$

Furthermore, for each 0 $< _c$ *the measure has a finite exponential moment: there exists* $( ) > 0$ *such that*

$$E \; [e^{\; (0)}] < 1:$$

Proof. Now to prove this we consider separately two di erent cases. Assume first that Z is defined for all positive reals, namely the radius of convergence is infinite. Suppose, by contradiction, that the function is bounded by some constant C_0. Then from equation 5.3 it would follow

$$@ \; \log Z( ) \quad C_0 \quad ^1:$$

Then, by integrating over ϕ we get that for every $\phi > 1$,

$$Z(\phi) \geq Z(1)\,\phi^{C_0}.$$

However, this is in contradiction with the fact that

$$Z(\phi) \geq \frac{\phi^k}{g!(k)}$$

for every integer k by the definition of Z.

Let us assume now, for the second part, that the radius of convergence is finite. Fix some positive $\phi_0 < \phi_c$. Since $Z(\phi)$ is a smooth increasing function, for $\phi \geq \phi_0$,

$$\log Z(\phi) \leq \log Z(\phi_0) + \frac{1}{\phi_0}\int_{\phi_0}^{\phi} \psi\,\partial_\psi \log Z(\psi)\,d\psi.$$

Since, on the other hand,

$$R(\phi) = \phi\,\partial_\phi \log Z(\phi),$$

it follows that

$$\phi_0 \log\frac{Z(\phi)}{Z(\phi_0)} \leq \int_{\phi_0}^{\phi} R(\psi)\,d\psi.$$

Since the left hand side of this inequality, by assumption 5.1, increases to ∞ as $\phi \to \phi_c$, we obtain that

$$\lim_{\phi\to\phi_c}\int_0^{\phi} R(\psi)\,d\psi = \infty.$$

Since the function R is increasing the first statement of the lemma is proved.

Lastly, notice that

$$E_{\nu_\phi}[e^{\theta\eta(0)}] = \frac{Z(\phi e^{\theta})}{Z(\phi)}.$$

Therefore, the second statement follows from assumption 5.1. □

At this point we will give an example of zero range dynamics that does not possess an invariant product measure for each density $\rho \geq 0$. However, because of the previous lemma, the partition function $Z(\phi)$ cannot satisfy the assumption 5.1.

Example 1. Consider a one-dimensional, nearest neighbor, symmetric zero range process, that is $p(-1) = p(1) = 1/2$, with jump rate $g(k) = (1+k^{-1})^3$ for $k \geq 1$. Then, $\phi_c = 1$ and the partition function is

$$Z(\phi) = 1 + \sum_{k\geq 1}\frac{\phi^k}{(k+1)^3}$$

so that

$$\lim_{\varphi \to 1} Z(\varphi) = 1 + \sum_{k \ge 1} \frac{1}{(k+1)^3} < \infty .$$

Consider an invariant product measure μ. From proposition 4, since μ is invariant, we have that $\int L\eta(x) d\mu = 0$ for every x. Denote by ρ_x the expectation of $g(\eta(x))$ under μ: $\rho_x = E_\mu[g(\eta(x))]$. Since $L\eta(x) = (1/2)(g(\eta(x+1)) + g(\eta(x-1)) - 2g(\eta(x)))$, the previous identity gives that $(\Delta_L \rho)_x = 0$, if Δ_L stands for the discrete Laplacian. This identity forces ρ_x to be constant; say equal to φ.

On the other hand, for every $x \in \mathbb{T}_L^d$ and $\alpha > 0$, $\int L\mathbf{1}\{\eta(x) \ge \alpha\} d\mu = 0$. Since

$$\begin{aligned} L\mathbf{1}\{\eta(x) \ge \alpha\} = & -g(\alpha)\mathbf{1}\{\eta(x) = \alpha\} \\ & + (1/2)\mathbf{1}\{\eta(x) = \alpha - 1\}(g(\eta(x+1)) + g(\eta(x-1))); \end{aligned}$$

since the measure μ is assumed to be product and since $E_\mu[g(\eta(x))] = \varphi$ is constant, we have that

$$g(\alpha)\,\mu\{\eta;\ \eta(x) = \alpha\} = \varphi\,\mu\{\eta;\ \eta(x) = \alpha - 1\}.$$

Furthermore, in this example, since $g(k) = (1 + k^{-1})^3$,

$$R(\varphi) = \sum_{k \ge 1} k \frac{\varphi^k}{(k+1)^3}$$

so that

$$\lim_{\varphi \to 1} R(\varphi) = \sum_{k \ge 1} \frac{k}{(k+1)^3} = \rho_c < \infty .$$

Consequently, for $\rho > \rho_c$, there is no invariant product measure with density ρ.

5.3 Relation to the Simple Exclusion Process

There exists an exact mapping from the one-dimensional zero-range process to the one-dimensional simple exclusion process. This is illustrated in the figure below. The idea is to consider the particles of the zero range process as the zeros (empty sites) of the exclusion process. Then the sites of the zero range process become the moving particles of the exclusion process. This is possible because of the preservation of the order of particles under the simple exclusion dynamics. Note that in the exclusion process we will have L particles hopping on a lattice of $L + N$ sites.

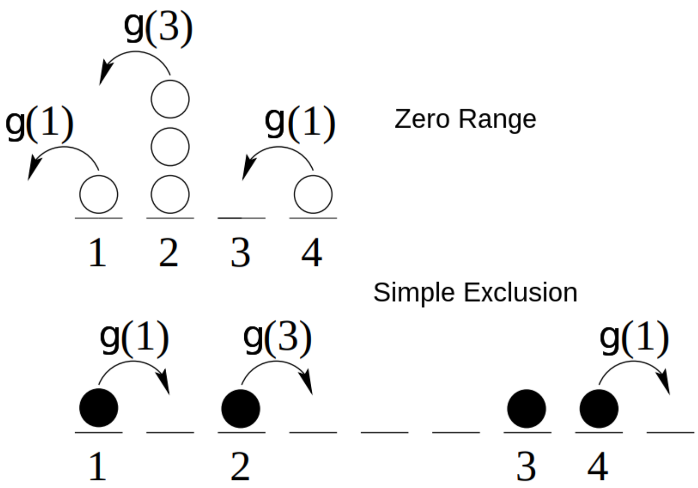

Figure 5.1: Equivalence of zero range process and simple exclusion process

An interesting feature of the mapping is that it converts a model where the local degree of freedom can take unbounded values (particle number in the zero range process) to a model where the local site variable is restricted to two values. On the other hand, a hopping rate $g(k)$ which is dependent on k corresponds to a hopping rate in the exclusion process which depends on the gap to the particle in front. Thus, the particles can "feel" each other's presence and it is possible to have a long-range interaction.

5.4 Supercritical Properties

Throughout this section we will see properties of zero range processes where $\rho_c < \infty$. As a result, there is a critical background density and excess particles condense on a non-extensive fraction of the volume. Precisely, if $\rho > \rho_c$ then

$$\lim_{\substack{N,L\to\infty \\ N/L\to\rho}} \mu^{N,L}\left(\max_{x\in \mathbb{T}_L^d} \eta(x) - (\rho-\rho_c)L\right) = 1.$$

While, in the subcritical case, the size of the largest component is of order $\log(L)$ [9]. If we were to get the picture of the system in the supercritical phase, we would distribute the bulk of the sites according to independent copies of ν_{ρ_c} and pile all the excess mas on a single randomly located site.

We will now study the case where $g(k)$ is decreasing in k, which then induces an effective attraction between particles. Specifically, the jump rates will be given by

$$g(k) = \begin{cases} 0, & \text{if } k = 0 \\ 1 + \frac{b}{k}, & \text{if } k \geq 1 \end{cases}$$

as studied in [3]. The author, also, observed for $b > 2$ that $\lim_{\lambda \to \lambda_c} Z(\lambda)$ and ρ_c are both finite. With this choice of g, we also get

$$g!(k) = \frac{(1+b)_k}{k!} = \frac{\Gamma(b+k+1)}{\Gamma(b+1)k!} \sim \frac{k^b}{\Gamma(b+1)};$$

where $(\lambda)_k = \prod_{i=0}^{k-1}(\lambda + i)$ denotes the Pochhammer symbol and $\Gamma(\cdot)$ denotes the standard Gamma function. The grand canonical partition function is

$$Z(\lambda) = \sum_{k=0}^{\infty} \frac{(1)_k (1)_k}{(1+b)_k} \frac{\lambda^k}{k!}$$

and its radius of convergence is $\lambda_c = 1$.

At this point, we will analyze the grand-canonical single site measure $\nu = \nu_\lambda$ in the limit $\lambda \to 1$, namely near the critical density ρ_c. For $\rho_c < \infty$ the limit ν_1 is well defined and it is the distribution of the non-condensed phase for super critical systems with $N/L = \rho > \rho_c$. As long as $\lambda < 1$ the distribution ν_λ has exponential moments. For $\lambda = 1$ the exponential tail of ν_λ disappears and the tail becomes proportional to $1/g!(k)$. These distributions have moments up to order $b - 1$. Therefore, different scenarios are encountered as b is varied.

For $b \le 1$ we get that

$$Z(\lambda) \to \infty;$$
$$\rho(\lambda) \to \rho_c = \infty;$$

as $\lambda \to 1$. For every density, the invariant distribution in the limit $L \to \infty$ is given by the grand-canonical measure ν_λ. The probability to have a fixed number of particles on a given site vanishes with increasing density. Thus in the limit there is an infinite number of particles on every site with probability one, as it should be for homogeneous systems with $\rho \to \infty$.

For $1 < b \le 2$, as we see a change of order, we get that

$$Z(\lambda) \to Z(1) = \frac{b}{b-1};$$
$$\rho(\lambda) \to \rho_c = \infty;$$

as $\lambda \to 1$. In particular, $\rho_c = \infty$ and the stationary distribution is described by the grand-canonical ensemble for every density ρ.

However, the character of this distribution for large ρ differs from the previous case, where $b \le 1$. Since $Z(1)$ is finite, ν_1 is well defined and there is a positive probability to have a fixed number of particles at a given site,

$$\nu_1(0) = \frac{1}{Z(1)} = \frac{b-1}{b};$$

$$\nu_1(k) = \frac{1}{Z(1)g!(k)} \sim \Gamma(b)(b-1)k^{-b} \text{ for large } k.$$

For example, the probability of an empty site, given by

$$\pi(0) = \frac{1}{Z(\phi(\rho))},$$

decreases monotonically. Note that in case $b \leq 1$ the probability vanishes in the limit $\rho \to \infty$, while here, it reaches the non-zero value $\pi_1(0) = (b-1)/b$. So no matter how large the density, the fraction of empty sites in a typical configuration is always greater than $(b-1)/b$.

For $b > 2$, besides the normalization also the first moment of the grand canonical distribution converges:

$$Z(\phi) \to Z(1) = \frac{b}{b-1},$$

$$\rho(\phi) \to \rho_c = \frac{1}{b-2},$$

as $\phi \to 1$. In addition, for $b > 3$ also the second moment σ^2 of the distribution π_1 is finite, with

$$\sigma^2 = \frac{(b-1)^2}{(b-2)^2(b-3)},$$

and the number of particles satisfies the usual central limit theorem

$$\frac{\sum \eta(x) - \rho_c L}{\sigma\sqrt{L}} \to N(0,1).$$

One more thing we should note is that the invariant distribution investigated so far carries no information on the dynamics of the condensation. A natural initial condition is to start with particles uniformly distributed at the supercritical density $\rho > \rho_c$. In the beginning the excess particles condense at a few random sites. Such a site containing many excess particles is called a cluster site. On the remaining sites, called bulk sites, the distribution relaxes to π_1. With increasing time the larger clusters will gain particles at the expense of the smaller ones, causing some of the clusters to disappear. Eventually only a single cluster containing all excess particles survives, which is typical for the invariant distribution, as was discussed starting this section. We will observe the above image later in our simulations.

Chapter 6

Experimental Results

In this chapter we will show all the results we acquired from our experiments on the subject. We have simulated models of Interacting Particle Systems that we studied in the previous chapters and furthermore, we have tried to determine various scaling limits. In addition, we give an efficient method for simulating such Interacting Particle Systems.

6.1 Independent Random Walks

The dynamics of independent random walks were specified in section 3.1. Moreover, in section 3.2, it was proved that for a system of indistinguishable particles following random walks, there is a unique family of invariant measures, parametrized by the density of particles, called the Poisson measure.

6.1.1 Invariant Distribution

First, let us validate the result of the invariant distribution. We consider an one-dimensional torus with 1000 sites. Then, we distribute the particles initially according to the Poisson measure and we observe the distribution at later times of order N and N^2. In the following figures, we see, for different particle densities, that this result, indeed, stands.

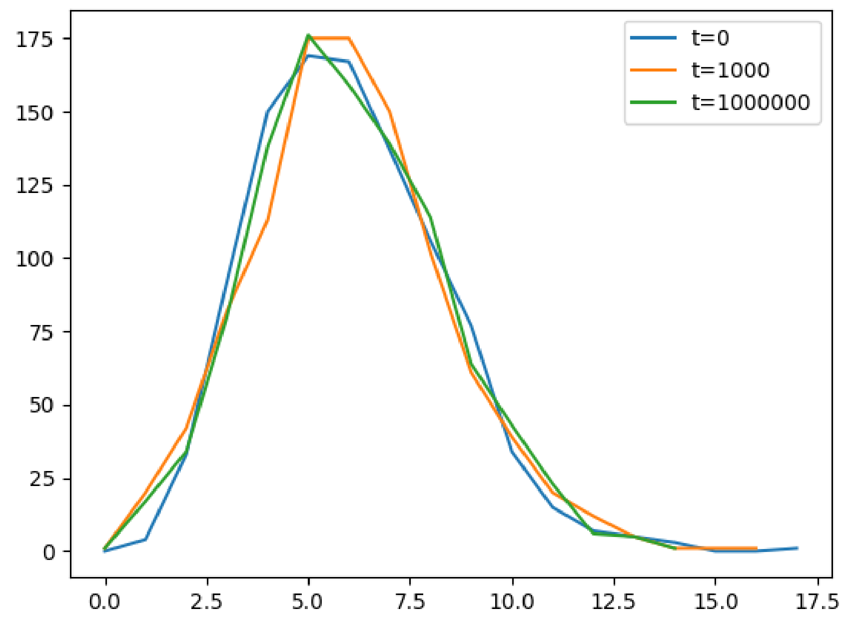

Figure 6.1: Distribution at time t of IRW on T^1_{1000} with = 5

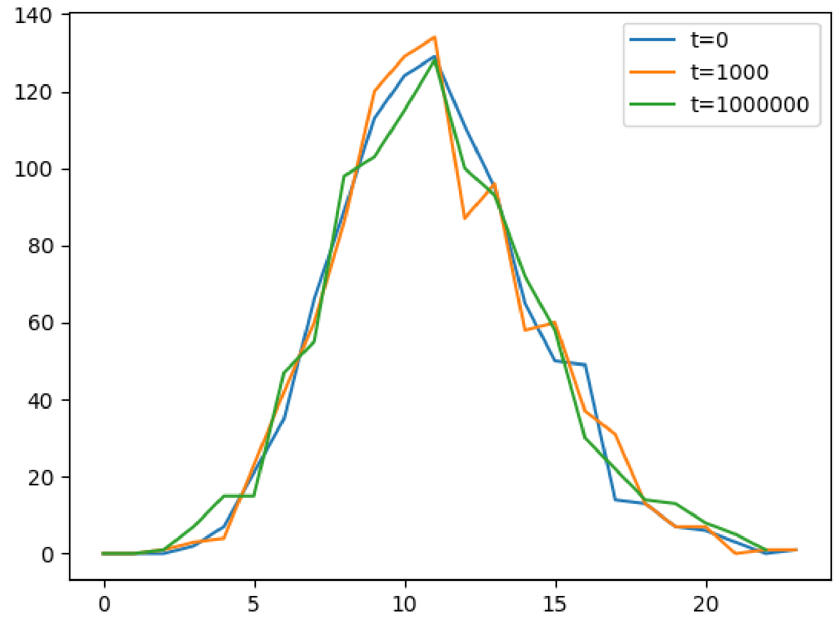

Figure 6.2: Distribution at time t of IRW on T^1_{1000} with = 10

6.1.2 2D Visualization

In addition, we wanted to visualize the evolution of a system of independent random walks in two dimensions. In order to achieve that, we have considered an image which represent the two-dimensional torus and we created a logarithmic color scale for the number of particles in each site. According to the normalized logarithmic value of the number of particles, the color is outputted in the following spectre:

$$\text{white} \to \text{yellow} \to \text{red} \to \text{black}. \tag{6.1}$$

In the images placed below, we see the evolution of the system, with total density 10, after some time, with the particles initially positioned at the center of the torus. In the first image, the elementary transition probability $\{p(x,y) : (x,y) \in T^2_{100}\}$ is symmetric, when in the second is asymmetric with:

$$p(1,0) = 0.3 \text{ (right)}, \; p(-1,0) = 0.1 \text{ (left)},$$

$$p(0,1) = 0.5 \text{ (up)}, \; p(0,-1) = 0.1 \text{ (down)}.$$

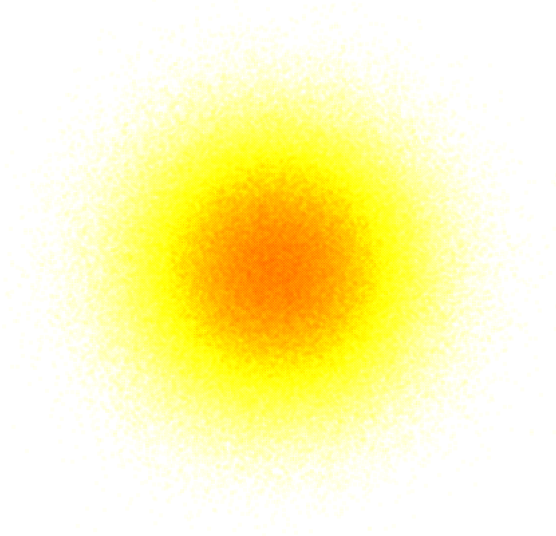

Figure 6.3: Symmetric IRW on T^2_{100}

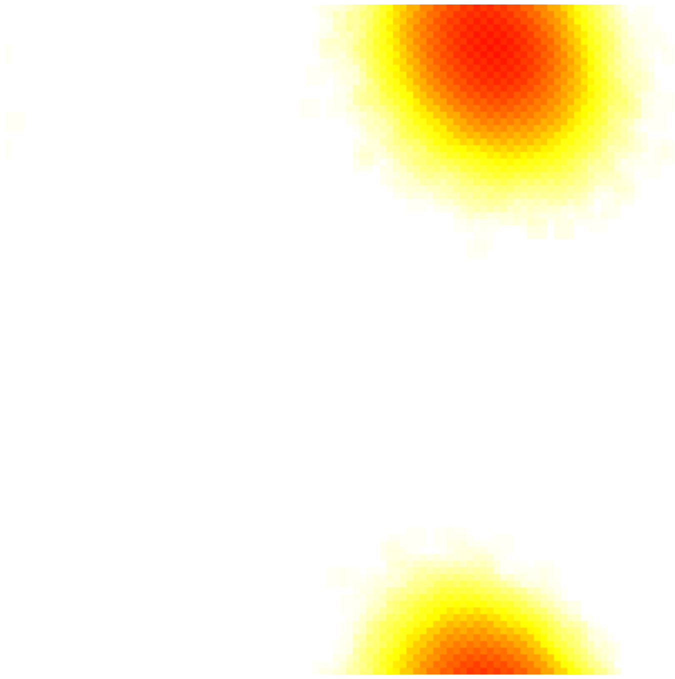

Figure 6.4: Asymmetric IRW on T^2_{100} at time $t_1 > 0$

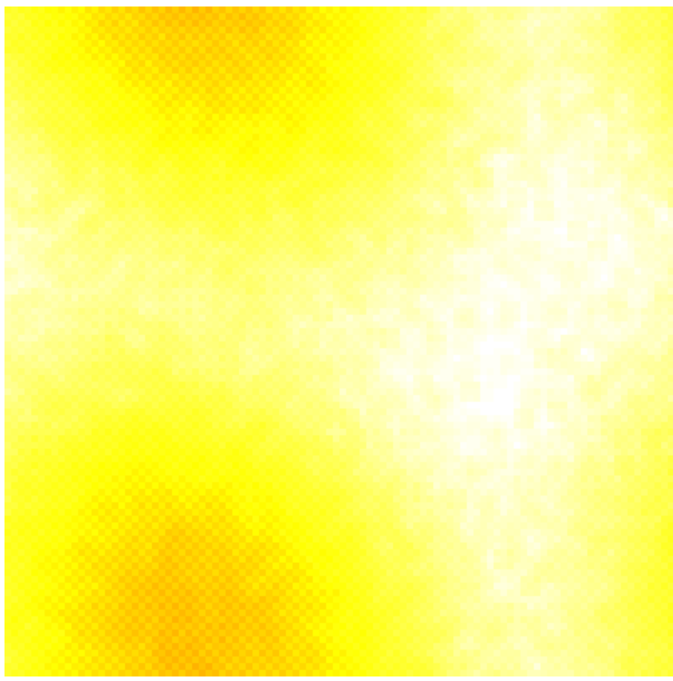

Figure 6.5: Asymmetric IRW on T^2_{100} at time $t_2 > t_1$

6.2 Continuous to Discrete Time Simulations

In this section we see an e□cient method for simulating particle systems with interaction. This method was not needed for the previous model, as interaction does not actually exists. Throughout the previous chapters we define the time evolution of our systems to be continuous and that is something necessary in order to support other concepts like scaling limits. However, as you know, in a computer, the evolution must happen in discrete times.

In section 2.2 we said that a Markov chain is a Markov process $(\ _t)_{t \ 0}$ defined on a countable set and it is characterized by the transition rates $c(\ ; \ ^0) \quad 0$. Furthermore, we denoted by

$$W \ = \inf f t \quad 0 : \ _t \neq \ g$$

the holding time in state , and by

$$c \ = \sum_{\ ^0 \neq \ } c(\ ; \ ^0):$$

the total exit rate out of state . Obviously if $c \ = 0$, then $W \ = 1$. After that, we proved proposition 1, which certifies that if $c \ > 0$, then

$$W \quad Exp(c \ );$$

where $Exp(\ )$ denotes the exponential distribution with rate .

While simulating a system of particles with interaction on a torus $\mathbb{T}^d_L$, there is always a large loop, in which each iteration represents a transition, namely the jump of one particle. The choice of the particle is strongly connected with the holding time in its state. Note here that the transition rates for the zero range process are:

$$c(\ ; \ ^{x;y}) = g(\ (x))p(y \quad x)\frac{1}{\ (x)}:$$

With this notation, the steps of the simulation of zero range processes (similarly in SEP) are expressed by the following algorithm.

```
rates    sum{g( (x)) for each x in T^d_L}
time    0
while true do
  choose a site x with probability g( (x))/rates
  choose a site y with probability p(y   x)
   (x)      (x)   1
   (y)      (y) + 1
  time    time + Exp(rates)
  rates    g( (x)) + g( (y))   g( (x) + 1)   g( (y)   1)
end while
```

We believe that all the steps are pretty straight-forward. Remember remark 1, where we proved that

$$\min(Exp(\lambda_1);\dots;Exp(\lambda_n)) \sim Exp(\sum_{i=1}^{n} \lambda_i).$$

This justifies the increments in *time* variable.

6.3 Simple Exclusion Process

The dynamics of simple exclusion process were specified in section 4.1. Furthermore, in section 4.2, it was proved that for a system of interacting particles following simple exclusion process, there is a unique family of invariant measures, parametrized by the density of particles, called the Bernoulli measure.

Later in that chapter, in section 4.3, we presented the nearest neighbor asymmetric simple exclusion process with step initial condition. We will now perform experiments regarding the behavior of the distance $|X_1|$ that the marginal particle has covered on a given time.

First, we will establish the dependence of the mean value of $|X_1(t)|$ from the time t, for both totally asymmetric and non-totally asymmetric cases. We assume that

$$E(|X_1(t)|) \sim ct^{\alpha},$$

which is expressed linearly

$$\log E(|X_1(t)|) \sim \alpha \log t + \log c.$$

For the totally asymmetric case we found that

$$E(|X_1(t)|) \approx t.$$

Now, remember the notation $\gamma = q - p$, where p is the probability that a particle jumps one place at the right and q that jumps one place at the left. When in the totally asymmetric case is $\gamma = 1$, we also consider the asymmetric case where $\gamma = \frac{3}{4} - \frac{1}{4} = \frac{1}{2}$. The result in this case was

$$E(|X_1(t)|) \approx \frac{t}{2} = \gamma t.$$

Finally, one could say that

$$E(|X_1(t/\gamma)|) \approx t$$

for every $\gamma \in (0;1]$.

Next, we need to establish the order of the fluctuations around the mean value as the time increases. Following the same strategy as before we found for the totally asymmetric case that

$$|X_1(t)| \quad E(|X_1(t)|) \quad O(\sqrt{t}).$$

Now regarding the fluctuations for the non-totally asymmetric version we found that

$$|X_1(t)| \quad E(|X_1(t)|) \quad O(t^{0.6}).$$

And then, by looking at the following plot of the distribution of the translated and normalized value of $|X_1(t)|$ for the non-totally asymmetric case and the following table we notice an extraordinary similarity with the Tracy-Widom distribution.

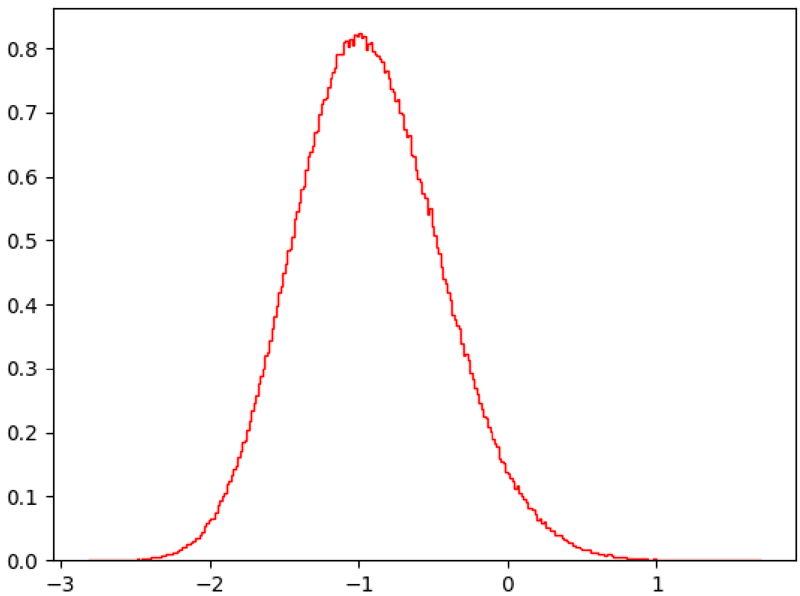

Figure 6.6: Distribution of $|X_1(t)|$ with $\quad = 0.5$

Mean	Variance	Skewness	Excess kurtosis
-0.93205	0.241756	0.23716	0.0714276

Table 6.1: Characteristics of the distribution

6.4 Zero Range Process

Throughout this section we will perform simulations on Zero Range Processes, considering the Evans model [3], where

$$g(k) = \begin{cases} 0, & \text{if } k = 0 \\ 1 + \frac{b}{k}, & \text{if } k \geq 1 \end{cases}$$

In addition, we will consider only supercritical cases with density $\rho > \rho_c$ and $b > 2$ to ensure that $\rho_c < \infty$.

The following image represents a two-dimensional torus equipped with the normalized logarithmic color scale described in 6.1. In order to produce it, we have let a zero range process, with particles uniformly initially distributed, unfold for a considerable amount of time t. Then each site in the picture represents the maximum number of particles accommodated in that site for some time $t' \leq t$. With that information we can roughly deduce all the system evolution from time 0 in the following way.

As we described in section 5.4, starting from initial condition with particles uniformly distributed at the supercritical density $\rho > \rho_c$, the excess particles condense at a few random sites, called cluster sites. With increasing time the larger clusters will gain particles at the expense of the smaller ones, causing some of the clusters to disappear. In that way, if you choose a threshold density $\rho_{th} > \rho_c$ and filter the image with it, then you get the cluster sites for some arbitrary time. And if you increase that density, then you get fewer cluster sites which correspond to some later time.

Figure 6.7: Zero Range Process with state $\eta_t(x) = \max_{0 \leq t' \leq t} \eta_{t'}(x)$

6.4.1 Scaling Order

Now, we will attempt to determine the order of the time needed for the system to reach equilibrium. We will work in one dimension. Remember here that the equilibrium state contains one condensed site and all the others distributed according to $_c$. Consider the following three initial distributions:

1. (x) = ; x 2 T_L
2. (0) = L 2 $_c$(L 1) and (x) = 2 $_c$; x 2 T_L=f 0g
3. (0) = L 4 $_c$L=5,
   (x) = 4 $_c$; x 2 [2L=5; 3L=5), otherwise (x) = 0

The last case is visualized as follows.

Figure 6.8: Example with L = 1000, p = 1 , p_c = 0:1

Although the actual time for reaching equilibrium from these states is di erent, the order of the time is the same. After running simulations for two di erent cases of elementary jump probability we found the following results. In a totally asymmetric zero range process, namely with p(1) = 1 , we have that

$$T_{eq} = O(L^2):$$

When in a symmetric zero range process, namely with p(1) = p(1) = 1 =2, we have that,

$$T_{eq} = O(L^3):$$

We would expect the time in the symmetric case to be of higher order as the particles di use without a drift.

6.4.2 Tagged Particles

Here we will assume the third initial distribution described above and we will study the dynamics of a tagged particle, whose location will be traced, in contrast with the others indistinguishable particles. Consider the following three classes of a tagged particle. If a particle is about to jump from the site x where the tagged particle is located,

> First class: then this particle will always be the tagged one,
>
> Random: then this particle will be the tagged one with probability 1= (x),
>
> Second class: then this particle will be the tagged one if it is the last one at the site.

We are called here to determine the dependence of the distance of the tagged particle from its initial position from the time t. The tagged particle will be initially located at site $x = L/2$. We also scale the space by L and the time by L^2 [15]. The simulations gave us:

> First class: $X_{tag}(tL^2)/L = O(1)$,
>
> Random: $X_{tag}(tL^2)/L = O(\sqrt{t})$,
>
> Second class: $X_{tag}(tL^2)/L = O(\sqrt{t})$.

We suspect that the first class tagged particles do not seem to depend on the time because of their tendency to reach fast the condensed site.

Chapter 7

Conclusion

Closing this diploma thesis, we would like do a short review of the ideas and problems studied and sum up its contribution. Furthermore, we will describe, in a bit, some possible future lines of work on the subject.

7.1 Contribution

We started by establishing all the essential mathematical foundations, derived from the eld of Stochastic Processes, in order to describe completely the notion of Interacting Particle Systems. This necessity has emerged due to the wide range of applications of Interacting Particle Systems in natural problems. The next step, was to de ne the simplest form of a particle system, namely the independent random walks, and then start asking questions about the underlying behaviors. These questions included the existence of invariant distributions in the system, the deduction of the macroscopic pro- le of the system in given time and space scales, de ning these scales, etc. After that, it was time to turn our attention to a bit more complex Interacting Particle Systems, like the Simple Exclusion Process and the Zero Range Process. These models along with many popular variances of them are already widely studied by the researchers. In one of the variances of Simple Exclusion Process, the asymmetric one with step initial condition, the Tracy-Widom distribution appears. Occasioned by that, follows a discussion regarding the universality that this speci c distribution seem to exhibit lately. In addition, while studying Zero Range Processes, we focused especially on a variance where attraction is introduced between the particles. Such model has started to be applied more and more often. Furthermore, the simulations which we performed gave us a very good intuitive understanding of our models and help us determine and validate several properties.

7.2 Future Work

As future lines of work on the field, we would suggest gaining a deeper understanding on the properties of Tracy-Widom distribution, as it seem to play a central role in concepts with interacting components. Moreover, it would be very advantageous to apply Variance Reduction Techniques on the simulations including tagged particles, as they could manifest considerably big fluctuations due to the nature of the quantity we want to approximate.

Sunoptik Ellhnik Ekdoq

Ta Sust mata Allhlepidr¸ntwn SwmatidÐwn eÐnai montèla pou sunant¸ntai se poll^ fusik^ sust mata, gia ta opoÐa oi topikoÐ mhqanismoÐ eÐnai polÔ aploÐ, all^ mporeÐ na eÐnai polÔ dÔskolo na exaqjeÐ mia kajolik sumperifor^. Stic perissìterec peript¸seic, oi par^gontec pou sumb^lloun sth duskolÐa aut eÐnai h eisagwg stoqastik c dunamik c kai allhlepÐdrashc se aut^. ParadeÐgmata mporoÔn na entopistoÔn se probl mata apì tic fusikèc epist mec, ìpwc sust mata di^qushc thc antÐdrashc kai swmatÐdia aerÐwn, pou epekteÐnontai se probl mata apì tic koinwnikèc epist mec, ìpwc h ro thc kukloforÐac, h dunamik thc gn¸shc kai h ex^plwsh twn epidhmi¸n.

O kÔrioc skopìc thc melèthc tètoiwn Susthm^twn Allhlepidr¸ntwn SwmatidÐwn eÐnai h exagwg thc makroskopik c sumperifor^c, h opoÐa sun jwc perigr^fetai apì udrodunamikèc exis¸seic, apì th mikroskopik allhlepÐdrash, dhlad apì touc upokeÐmenouc stoqastikoÔc mhqanismoÔc. O majhmatikìc ìroc pou susqetÐzetai me autì onom^zetai ìria klim^kwshc . Ta ìria klim^kwshc èqoun meg^lo endiafèron gia th fusik kai idiaÐtera gia ta majhmatik^. Epiplèon, mac endiafèrei sun jwc na broÔme katast^seic isorropÐac sto sÔsthm^ mac, dhlad ìtan to makroskopikì profÐl den all^zei. Autèc oi isorropÐec perigr^fontai majhmatik^ me ton qarakthrismì analloÐwtwn katanom¸n.

Anex^rthtoi TuqaÐoi PerÐpatoi

Se aut n thn enìthta jèloume na diereun soume to sÔsthma twn mh diaqwrÐsimwn swmatidÐwn pou ekteloÔn anex^rthtouc tuqaÐouc perip^touc. O kÔrioc stìqoc mac eÐnai o anagn¸sthc na exoikeiwjeÐ me basikèc ènnoiec kai teqnikèc pou qrhsimopoioÔntai ston tomèa twn Susthm^twn AllhlepÐdr¸ntwn SwmatidÐwn.

Montèlo

Shmei¸ste me Z^d to d-di^stato plègma akèraiwn arijm¸n. Gia èna jetikì akèraio L, shmei¸ste me T_L ton tìmo me L shmeÐa: $T_L = Z_L =$ f0; 1; :::; L 1g kai $T_L^d = (T_L)^d$. Ed¸ to L antiproswpeÔei to antÐstrofo thc apìstashc metaxÔ twn shmeÐwn T_L^d, dhlad twn perioq¸n twn swmatidÐwn, oi opoÐec a-ntiproswpeÔontai apì x, y kai z. Af nontac $L \to \infty$, h apìstash metaxÔ twn swmatidÐwn ja p˜ei sto mhdèn kai ètsi pern˜me apì to mikroskopikì sto makroskopikì pedÐo.

Jèloume na perigr˜youme thn exèlixh tou sust matoc, opìte af ste to N na dhl¸sei ton sunolikì arijmì twn swmatidÐwn kai af ste ta $x_1; x_2; \ldots; x_N$ na dhl¸soun tic arqikèc touc jèseic. EpÐshc, epeid ta swmatÐdia exelÐsso-ntai wc anex˜rthtoi tuqaÐoi perÐpatoi diakritoÔ qrìnou, prèpei na orÐsoume mia analloÐwth pijanìthta met˜bashc $p(x; y)$ sto Z^d, gia thn opoÐa $p(x; y) = p(0; y - x) =: p(y - x)$ gia k˜poia pijanìthta $p(\cdot)$ sto Z^d, pou onom˜zetai stoi-qei¸dhc metabatik pijanìthta tou sust matoc. Aut h pijanìthta ekfr˜zei to stoqastikì qarakthristikì tou tuqaÐou perip˜tou.

'Estw ìti to $p_t(x; y)$ antiproswpeÔei thn pijanìthta na eÐnai èna swmatÐdio sto q¸ro y sto qrìno t gia mia diakrit qronik tuqaÐa poreÐa me stoiqei¸dh pijanìthta met˜bashc $p(\cdot)$ xekin¸ntac apì to x. Epiplèon, to $p_t(\cdot\,;\cdot)$ klhro-nomeÐ thn idiìthta met˜bashc apì to $p(\cdot\,;\cdot)$ kai ètsi $p_t(x; y) = p_t(0; y - x) =: p_t(y - x)$.

EÐnai kairìc na perigr˜youme thn kÐnhsh k˜je swmatidÐou. Ac p˜roume N anex˜rthtec metablhtèc tuqaÐwn perip˜twn $\{Z_t^1; Z_t^2; \ldots; Z_T^N\}$ sto Z^d me stoi-qei¸dh pijanìthta met˜bashc $p(\cdot)$ kai arqik˜ sto mhdèn. Wc apotèlesma, h jèsh k˜je swmatidÐou i ston tìmo T_L^d sto qrìno t ja eÐnai

$$X_t^i = x_i + Z_t^i \mod L:$$

Wstìso, epeid ta swmatÐdia den diakrÐnontai sto montèlo mac, den mac en-diafèrei h akrib c jèsh k˜je swmatidÐou, all˜ o arijmìc twn swmatidÐwn se k˜je jèsh tou q¸rou. Sugkekrimèna, o q¸roc kat˜stashc tou sust matoc, pou onom˜zetai epÐshc q¸roc diamìrfwshc, eÐnai $\mathbb{N}^{T_L^d}$. Oi katast˜seic shmei-¸nontai me η, ξ kai ζ. SÔmfwna me autìn ton orismì, e˜n to x eÐnai q¸roc sto T_L^d, tìte $\eta(x)$ eÐnai o arijmìc twn swmatidÐwn autoÔ tou q¸rou gia thn kat˜stash η. Epomènwc, an ta swmatÐdia eÐnai arqik˜ sta $x_1; x_2; \ldots x_N$, tìte

$$\eta(x) = \sum_{i=1}^{N} 1\{x = x_i\}:$$

Apì thn ˜llh pleur˜, e˜n mac d¸soune $(\eta(x) : x \in T_L^d)$, mporoÔme pr¸ta na epishm˜noume ta swmatÐdia kai na ta af soume na exeliqjoÔn sÔmfwna me thn stoqastik dunamik pou perigr˜yame.

Fusik˜, jèloume thn diamìrfwsh se qrìno t, h opoÐa ja sumbolÐzetai me $\eta_t(x)$ kai ja oristeÐ apì

$$\eta_t(x) = \sum_{i=1}^{N} 1\{x = X_t^i\}:$$

Epiplèon, h diadikasÐa $(\ _t)_{t\ 0}$ klhronomeÐ thn idiìthta Markov apì touc tuqaÐouc perip˜touc X^i_t; 1 i N g epeid ìla ta swmatÐdia èqoun thn Ðdia stoiqei¸dh pijanìthta met˜bashc kai den allhlepidroÔn metaxÔ touc.

Mètro Poisson

Dedomènou ìti o q¸roc kat˜stashc eÐnai peperasmènoc kai dedomènou ìti o sunolikìc arijmìc twn swmatidÐwn eÐnai h monadik posìthta pou diathreÐtai apì th dunamik tou sust matoc, gia k˜je jetikì akèraio N pou antiprosw-peÔei ton sunolikì arijmì swmatidÐwn, up˜rqei mìno èna amet˜blhto mètro. H upost rixh thc stoiqei¸douc pijanìthtac met˜bashc p( ) par˜gei tp Z^d, dhlad h diadikasÐa eÐnai mh upobib˜simh. Ta mètra tou Poisson sth melèth mac ja paÐxoun kentrikì rìlo.

JumhjeÐte ìti h katanom Poisson me par˜metro > 0 eÐnai to mètro pijanìthtac ston N me

$$p_k = e \quad \frac{k}{k!}; \ k \ 2 \ N$$

kai o metasqhmatismìc Laplace eÐnai Ðsoc me

$$e \sum_{k=0}^{1} e^{\ k} \frac{k}{k!} = e \quad e^{e} \quad = e^{\ (e \quad 1)};$$

gia k˜je > 0.

OrÐzoume to parak˜tw mètro. Gia mia jetik sun˜rthsh : T^d_L ! R_+, onom˜zoume to Poisson mètro sto T^d_L pou sqetÐzetai me th sun˜rthsh , mia pijanìthta ston q¸ro diamìrfwshc $N^{T^d_L}$, pou upodhl¸netai apì $^L_{(\ )}$, èqontac tic parak˜tw dÔo idiìthtec.

- k˜tw apì to $^L_{(\ )}$ oi tuqaÐec metablhtèc ((x) : x 2 T^d_L), pou antipro-swpeÔoun ton sunolikì arijmì swmatidÐwn, prèpei na eÐnai anex˜rthtec,

- gia k˜je q¸ro x 2 T^d_L, h (x) katanèmetai sÔmfwna me mia katanom Poisson paramètrou (x).

Sthn perÐptwsh ìpou h sun˜rthsh eÐnai stajer kai Ðsh me , upodh-l¸noume to $^L_{(\ )}$, mìno me ^{L}. EpÐshc, ja upodhl¸noume thn anamenìmenh tim se sqèsh me èna mètro wc E .

To mètro $^L_{(\ )}$ qarakthrÐzetai apì ton poludi˜stato metasqhmatismì La-place:

$$E_{\ ^L_{(\ )}} \ e^{\ \sum_{x \, 2 \, T^d_L} (x) \ (x)} = \prod_{x \, 2 \, T^d_L} e^{\ (x)(e^{\ (x)} \ 1)} = e^{\ \sum_{x \, 2 \, T^d_L} e^{\ (x)(e^{\ (x)} \ 1)}}$$

gia k˜je jetik akoloujÐa ((x) : x 2 T^d_L) [4].

T¸ra, ac proqwr soume sto na diapist¸soume ìti ta mètra Poisson pou sqetÐzontai me stajerèc sunart seic eÐnai analloÐwta gia èna sÔsthma anex˘rthtwn tuqaÐwn perip˘twn.

Prìtash. An ta swmatÐdia arqik˘ katanemhjoÔn sÔmfwna me èna mètro Poisson pou sqetÐzetai me mia stajer sun˘rthsh Ðsh me tìte h katanom sto qrìno t eÐnai akrib¸c to Ðdio mètro Poisson [11]

Apìdeixh. OrÐste wc P_L to mètro pijanìthtac sto q¸ro monopati¸n $_L = N^{T^d_L} \quad N^{T^d_L}$, dhlad o q¸roc thc $(\ _t)_{t \quad 0}$ pou par˘getai apì tic dunamikèc twn anex˘rthtwn tuqaÐwn perip˘twn kai to arqikì mètro ^{L}. H anamenìmenh tim wc proc to P_L upodhl¸netai apì E_L. Se autì to shmeÐo, parathr ste th diafor˘ metaxÔ E_L, pou eÐnai h anamenìmenh tim se sqèsh me to mètro pou orÐzetai sto $N^{T^d_L}$ kai E_L, h opoÐa eÐnai h anamenìmenh tim se sqèsh me to mètro pou orÐzetai sto q¸ro monopati¸n $_L$. EÐnai eÔkolo na deic ìti

$$E_L[F(\ _0)] = E_L[F(\ )]$$

gia k˘je fragmènh suneq sun˘rthsh F sto $N^{T^d_L}$.

Dedomènou ìti to mètro $^L_{(\ )}$ qarakthrÐzetai apì ton poludi˘stato metasqhmatismì Laplace, ja upologÐsoume ed¸ thn anamenìmenh tim

$$E_L \quad e^{\ P_{x 2 T^d_L} \ (x) \ _t(x)}$$

gia k˘je jetik akolouja $(\ (x) : x \ 2 \ T^d_L)$. Epiplèon, gia ènan q¸ro y 2 T^d_L, ja upodeÐxoume me $X^{y;k}_t$ th jèsh sto qrìno t tou k-ostoÔ swmatidÐou xekin¸ntac apì y. Me autì ton trìpo, o arijmìc swmatidÐwn ston q¸ro x sto qrìno t ja eÐnai:

$$_t(x) = \sum_{y 2 T^d_L} \ \sum_{k=1}^{(y)} 1 f x = X^{y;k}_t g:$$

Kai tìte, anastrèfontac th seir˘ ajroÐsewc, ja p˘roume:

$$\sum_{x 2 T^d_L} (x) \ _t(x) = \sum_{y 2 T^d_L} \ \sum_{k=1}^{(y)} (X^{y;k}_t):$$

Dedomènou ìti k˘je swmatÐdio exelÐssetai anex˘rthta kai o sunolikìc arijmìc swmatidÐwn se k˘je jèsh sto qrìno 0 katanèmetai sÔmfwna me mia katanom Poissonparamètrou ,

$$E_L\, e^{\sum_{x \in T_L^d} (x)\ _t(x)} = E_L\, e^{\sum_{y \in T_L^d} \sum_{k=1}^{0(y)} (X_t^{y;k})} =$$

$$\prod_{y \in T_L^d} E_L\, e^{\left[\sum_{k=1}^{0(y)} (X_t^{y;k})\right]} =$$

$$\prod_{y \in T_L^d} \int {}^L(d\ )\ E\, e^{\left[(X_t^{y;1})\right]\ 0(y)} =$$

$$\prod_{y \in T_L^d} e^{\ E\ e^{(X_t^{y;1})}\ 1} = \prod_{y \in T_L^d} e^{\left(E\left[e^{(y+X_t)}\right]\ 1\right)}$$

'Opou X_t eÐnai ènac tuqaÐoc perÐpatoc sto qrìno ston tìmo T_L^d xekin¸ntac apì thn arq kai me pijanìthta met¨bashc $p_t^L(\ )$ me

$$p_t^L(x;y) = \sum_{z \in Z^d} p_t(x; y + Lz)$$

gia $x; y \in T_L^d$. Dedomènou ìti, ex orismoÔ,

$$E\left[e^{(y+X_t)}\right] = \sum_{x \in T_L^d} p_t^L(x\ \ y) e^{(x)};$$

tìte, antistrèfontac th seir¨ ajroÐsewc kai ìti $\sum_{y \in T_L^d} p_t^L(x\ \ y) = 1$, paÐrnoume

$$E_L\, e^{\sum_{x \in T_L^d} (x)\ _t(x)} = e^{\sum_{x \in T_L^d} \left(e^{(x)}\ 1\right)}:$$

Ja prèpei epÐshc na shmei¸soume ìti oi katanomèc Poisson eÐnai tètoiec ¸ste h anamenìmenh tim touc na isoÔtai me

$$\sum_{k\ 0} e\ \frac{k}{k!} k = \ :$$

Ta mètra Poisson me autìn ton trìpo fusiologik¨ parametropoioÔntai apì thn puknìthta twn swmatidÐwn. Epiplèon, apì ton nìmo twn meg¨lwn arijm¸n,

$$\lim_{L \to \infty} \frac{1}{|T_L^d|} \sum_{x \in T_L^d} (x) =$$

kat¨ pijanìthta se sqèsh me ^{L}. Sunep¸c, fantasteÐte ìti h par¨metroc perigr¨fei th mèsh puknìthta twn swmatidÐwn se èna "meg¨lo koutÐ".

Sumperasmatik¨, gia aut thn enìthta, diapist¸same thn Ôparxh miac oikogèneiac analloÐwtwn mètrwn miac paramètrou pou prosdiorÐzei thn puknìthta twn swmatidÐwn, h opoÐa eÐnai h monadik posìthta pou diathreÐtai apì thn exèlixh tou qrìnou.

Udrodunamik Sumperiforˆ

EÐpame prohgoumènwc ìti ènac apì touc kÔriouc stìqouc mac eÐnai na sunagˆgoume th makroskopik sumperiforˆ enìc sust matoc allhlepidr¸ntwn swmatidÐwn. Sthn perÐptwsh aut , h metˆbash apì ton mikrìkosmo se ston makrìkosmo ja gÐnei me thn pragmatopoÐhsh enìc orÐou sto opoÐo h apìstash metaxÔ twn jèsewn twn swmatidÐwn ja ftˆsei sto mhdèn. Autì den eÐnai dÔskolo na ekfrasteÐ kai eÐnai mia teqnik pou qrhsimopoieÐtai se polloÔc tomeÐc twn Majhmatik¸n.

An fantastoÔme ìti o diakritìc tìmoc T^d_L eÐnai enswmatwmènoc se ènan suneq tìmo $T^d = [0;1)^d$, dhlad paÐrnontac to plègma T^d me {korufèc} sta $x=L$; $x \in T^d_L$, tìte oi apostˆseic metaxÔ twn swmatidÐwn eÐnai $1=L$ kai teÐnoun sto mhdèn ìtan $L \to \infty$. Me autì ton trìpo, gia thn antÐstrofh qartogrˆfhsh, kˆje makroskopikì shmeÐo u sto T^d sundèetai me mia mikroskopik topojesÐa $x = [uL]$ sto T^d_L.

'Estw t¸ra $\rho_0 : T^d \to R_+$ na eÐnai mia omal sunˆrthsh pou perigrˆfei to arqikì profÐl puknìthtac. Sth sunèqeia, dianèmoume swmatÐdia sÔmfwna me èna mètro Poisson me argˆ metaballìmenh parˆmetro sto T^d, dhlad gia kˆje jetikì L kajorÐzoume thn parˆmetro thc katanom c Poisson ston q¸ro x na eÐnai Ðsh me $\rho_0(x=L)$.

Blèpoume ìti an xekin soume apì èna mètro Poisson me argˆ metaballìmenh parˆmetro tìte

$$E_{\nu^L_{\rho_0(\cdot)}}\left[e^{\sum_{x\in T^d_L}\lambda(x)\eta_t(x)}\right] = e^{\sum_{x\in T^d_L}\rho_0(x=L)\sum_{y\in T^d_L}p^L_t(y-x)\left(e^{\lambda(y)}-1\right)} = e^{\sum_{y\in T^d_L}\left(e^{\lambda(y)}-1\right)\sum_{x\in T^d_L}p^L_t(y-x)\rho_0(x=L)} =: e^{\sum_{y\in T^d_L}\left(e^{\lambda(y)}-1\right)\rho_{L;t}(y)}:$$

Sthn parapˆnw exÐswsh paÐrnoume ìti, se qrìno t, exakoloujoÔme na èqoume èna mètro Poisson me argˆ metaballìmenh parˆmetro, h opoÐa t¸ra eÐnai $\rho_{L;t}(\cdot)$ antÐ gia $\rho_0(\cdot=L)$.

Apì th rht morf tou $\rho_{L;t}$, èqoume ìti gia kˆje shmeÐo u sunèqeiac thc ρ_0,

$$\lim_{L\to\infty}\rho_{L;t}([uL]) = \rho_0(u):$$

Autì pou mac lèei h parapˆnw exÐswsh eÐnai ìti to profÐl parèmeine ametˆblhto. Parìlo pou o qrìnoc t èqei perˆsei, faÐnetai ìti to sÔsthma den eÐqe arketì qrìno gia na exeliqjeÐ kai autì antanaklˆ to gegonìc ìti se makroskopik klÐmaka ta swmatÐdia den kin jhkan. Me pijanìthta kontˆ sto 1, sth makroskopik klÐmaka, ta swmatÐdia se qrìno t èqoun kalÔyei apìstash tˆxhc L^{-1}.

Prokeimènou na epilujeÐ autì to prìblhma, prèpei na diakrÐnoume dÔo diaforetikèc qronikèc klÐmakec, kaj¸c èqoume dh diaforetikèc klÐmakec q¸rou, T^d kai $L^{1}T^d_L$. AntÐstoiqa, qreiazìmaste ènan mikroskopikì qrìnot kai ènan makroskopikì qrìno o opoÐoc ja eÐnai apeÐrwc meg˜loc se sqèsh me to t.

Upojètoume ìti h stoiqei¸dhc pijanìthta met˜bashc p( ) èqei peperasmènh anamenìmenh tim m := P xp(x) 2 R^d. Lème ìti o tuqaÐoc perÐpatoc eÐnai asÔmmetroc an m 6= 0, ìti eÐnai asÔmmetroc me mhdenikì mèso an h p( ) den eÐnai summetrik all˜ m = 0 kai ìti eÐnai summetrikìc an h p( ) eÐnai summetrik .

Sugkekrimèna, apì th rht èkfrash tou $_{L;tL}$ kai afoÔ upojèsame ìti to arqikì profÐl eÐnai omalì, èqoume

$$\lim_{L!1} \ _{L;tL}([uL]) = \ _0(u \quad mt) =: \quad (t;u)$$

gia k˜je u 2 T^d.

T¸ra mporeÐte na deÐte ìti me th nèa qronik klÐmaka, L, to profÐl den parèmeine amet˜blhto. AntÐjeta, parathroÔme èna nèo makroskopikì profÐl: to prwtìtupo pou metatopÐsthke apì mt.

'Ena sÔsthma swmatidÐwn pou exelÐssetai wc anex˜rthtoi asÔmmetroi tuqaÐoi perÐpatoi me peperasmènh pr¸th stigm se èna di˜stato tìmo èqei mia udrodunamik perigraf . H exèlixh tou profÐl puknìthtac perigr˜fetai apì th lÔsh thc diaforik c exÐswshc

$$@ \ + m \ r \quad = 0:$$

Wstìso, ìtan o tuqaÐoc perÐpatoc den eÐnai asÔmmetroc kai h anamenìmenh tim m exafanÐzetai, h lÔsh aut c thc diaforik c exÐswshc eÐnai stajer , pr˜gma pou shmaÐnei ìti to profÐl den ˜llaxe sthn qronik klÐmaka tL . Autì èqei nìhma diaisjhtik˜, kaj¸c to sÔsthma eÐnai t¸ra amerìlhpto proc thn kateÔjunsh. Wstìso, e˜n exet˜soume mia megalÔterh qronik klÐmaka, t˜xhc L^2, akìmh kai ìtan h mèsh ektìpish m exafanisteÐ, mporoÔme na parathr soume mia endiafèrousa exèlixh ston qrìno. [11]

Apl DiadikasÐa ApokleismoÔ

Se aut n thn enìthta mac endiafèroun oi diadikasÐec aploÔ apokleismoÔ. Autì to montèlo prot˜jhke sto [16] kai eÐnai èna apì ta aploÔstera kai pio melethmèna sust mata allhlepidr¸ntwn swmatidÐwn. H diadikasÐa aploÔ apokleismoÔ, se antÐjesh me touc anex˜rthtouc tuqaÐouc perip˜touc pou melet jhkan sthn prohgoÔmenh enìthta, epitrèpei to polÔ èna swmatÐdio an˜ q¸ro.

Pr¸ta ap 'ìla, o q¸roc kat˜stashc eÐnai f 0; 1g$^{T^d_L}$. Prokeimènou na apofeuqjeÐ h emf˜nish perissìterwn apì èna swmatidÐwn an˜ topojesÐa, eis˜goume ènan kanìna apokleismoÔ pou katastèllei k˜je ˜lma se dh kateqìmeno q¸ro. Sthn pragmatikìthta, ja epikentrwjoÔme mìno sthn aploÔsterh kathgorÐa diadikasi¸n apokleismoÔ: sust mata ìpou ta swmatÐdia phdoÔn, ìpote epitrèpetai to ˆlma, anexˆrthta apì ta ˆlla kai sÔmfwna me thn Ðdia stoiqei¸dh metabatik pijanìthta.

O genn torac

$$(Lf)(\;) := \sum_{x \in T_L^d} \sum_{z \in T_L^d} (x)(1 \quad (x+z))\, p^L(z)(f(\;^{x;x+z}) \quad f(\;));$$

ìpou $^{x;y}$ eÐnai katˆstash pou paÐrnoume apo thn af nontac èna swmatÐdio na metaphd sei apì to x sto y, dhlad

$$^{x;y}(z) = \begin{cases} (z) & \text{an } z \neq x; y, \\ (x) \quad 1 & \text{an } z = x, \\ (y) + 1 & \text{an } z = y \end{cases} \quad \text{kai} \quad p^L(z) := \sum_{y \in Z^d} p(z + yL);$$

orÐzei mia diadikasÐa Markov pou onomˆzetai apl diadikasÐa apokleismoÔ me stoiqei¸dh pijanìthta metabˆsewn p( ). Sth sugkekrimènh perÐptwsh ìpou p(z) = p(z) lème ìti eÐnai mia summetrik apl diadikasÐa apokleismoÔ.

PisteÔoume ìti h ermhneÐa eÐnai saf c. Metaxô t kai dt, kˆje swmatÐdio prospajeÐ, anexˆrthta apì ta ˆlla, na phd xei apì to x sto x + z me (ekjetikì) rujmì $p^L(z)$. To ˆlma katastèlletai eˆn odhgeÐ se dh kateqìmeno q¸ro.

UpenjumÐzoume ed¸ ìti mia diadikasÐa Markov lègetai ìti eÐnai mh upobibˆsimh an eÐnai dunatìn na ftˆsoume se opoiad pote katˆstash apì opoiod pote katˆstash. Epiplèon, dedomènou ìti h pijanìthta metˆbashc jewreÐtai ìti eÐnai peperasmènou eÔrouc, upˆrqei A_0 sto N tètoio ¸ste p(z) = 0 gia ìlec tic topojesÐec ektìc tou kÔbou $[\; A_0; A_0]^d$. Sugkekrimèna, ta $p^L(\;)$ kai p( ) sumpÐptoun gia L A_0. Gi ˆutìn ton lìgo, apì ed¸ kai pèra ja paraleÐyoume ton deÐkth L sthn stoiqei¸dh pijanìthta metˆbashc. Akìma, parathroÔme ìti o rujmìc me ton opoÐo èna swmatÐdio metaphdˆ apì to x sto y exartˆtai apì th diamìrfwsh mìno mèsw twn metablht¸n diamìrfwshc (x) kai (y). Aut h teleutaÐa exˆrthsh apì ta (x) kai (y) antikatoptrÐzei ton kanìna apokleismoÔ. Tèloc, parathr ste ìti o sunolikìc arijmìc swmatidÐwn diathreÐtai apì th dunamik .

Dhl¸noume me $= \;^L$, gia to 0 1, to mètro Bernoulli paramètrou , dhlad to analloÐwto mètro sto $f0; 1g^{T_L^d}$ me puknìthta . Sugkekrimèna, kˆtw apì , oi metablhtèc ((x) : x 2 T_L^d) eÐnai anexˆrthtec me

$$f \;\; (x) = 1\, g = \quad = 1 \qquad f \;\; (x) = 0\, g:$$

Prìtash. Ta mètra Bernoulli f : 0 1g eÐnai ametˆblhta gia aplèc diadikasÐec apokleismoÔ. Epiplèon, se sqèsh me kˆje , oi diergasÐec apokleismoÔ me pijanìthta stoiqei¸douc metˆbashc p(z) := p(z) eÐnai sumplhrwmatikèc me diadikasÐec me stoiqei¸dh pijanìthta pijanìthtac p(z). Sugkekrimèna, oi summetrikèc aplèc diadikasÐec apokleismoÔ eÐnai autosumplhrwmatikèc se sqèsh me kˆje . [11]

Apìdeixh. EĐnai eÔkolo na parathr sete ìti me mia apl allag twn metablht¸n

$$\int f(\ ^{0;z})g(\ )\ (0)(1\ \ (z))\ \ (d\ ) = \int f(\ )g(\ ^{0;z})\ (z)(1\ \ (0))\ \ (d\ ):$$

Aut h tautìthta, to gegonìc ìti 1 = $\sum_{z \in Z^d} p(z) = \sum_{z \in Z^d} p(\ z)$ kai mia allag sth seirˆ ajroĐsewc, apodeiknÔoun thn prìtash.

Se aut thn perĐptwsh, h oikogèneia twn analloĐwtwn mètrwn parametropoieĐtai apì thn puknìthta, afoÔ

$$E\ \ [\ (0)] = \ \ f\ \ (0) = 1\,g = \ :$$

AsÔmmetrh me bhmatik arqik katˆstash

Mia ekten¸c melethmènh parallag thc apl c diadikasĐac apokleismoÔ eĐnai h monodiˆstath plhsièsterh geitonik asÔmmetrh diadikasĐa apokleismoÔ me bhmatik arqik katˆstash. Se autìn ton tomèa, èqoun epiteuqjeĐ pollˆ shmantikˆ apotelèsmata apì touc Tracy kai Widom.

Ac exetˆsoume to plègma twn akeraĐwn Z. Sthn perĐptwsh bhmatik c arqik c katˆstashc, ta swmatĐdia ja arqĐsoun apì touc jetikoÔc akèraiouc Z_+. 'Opwc gnwrĐzete, èna swmatĐdio perimènei ekjetikì qrìno kai sth sunèqeia metakineĐtai proc ta dexiˆ me pijanìthta p an o sugkekrimènoc q¸roc eĐnai kenìc proc ta aristerˆ me pijanìthta q = 1 p an o sugkekrimènoc q¸roc eĐnai kenìc. Eˆn o q¸roc ston opoĐo prìkeitai na phd xei katalambˆnetai, tìte paramènei ekeĐ pou eĐnai.

H kÔria posìthta pou ja mac apasqol sei se aut thn enìthta eĐnai h jèsh tou m th swmatidĐou apì ta aristerˆ thn stigm t,

$$x_m(t),\ \text{me}\ x_m(0) = \ m:$$

Ed¸ ja upojèsoume epĐshc ìti p < q ¸ste na upˆrqei mia tˆsh proc ta aristerˆ. T¸ra ja xekin soume perigrˆfontac ta apotelèsmata pou parousiˆsthkan kai apodeĐqjhkan sto [21].

Oi suntˆktec sto [17], apèdeixan mia fìrmoula gia thn posìthta pou mac endiafèrei, pou isqÔei ìtan ta p kai q eĐnai mhdenikˆ. DĐnetai me bˆsh thn Fredholm orĐzousa enìc pur na $K(\ ;\ ^0)$ sto C_R, ènan kÔklo me kèntro sto mhdèn kai mia megˆlh aktĐna R. O pr¸toc tÔpoc eĐnai o akìloujoc

$$P(x_m(t=\ )\ \ x) = \int \frac{\det(I\ \ K\ )}{\prod_{k=0}^{m\ 1}(1\ \ ^k)}\frac{d}{\ }:$$

EÔkola prokÔptei apì thn parapˆnw exĐswsh ìti

$$P(x_1(t=\ ) > x\ ) = \det(\ I\ \ K\ ):$$

EĐnai xekˆjaro ìti to $P(x_m(t) > x\ ) = 0$ gia ìla ta t ìtan x m, kaj¸c gia èna swmatĐdio na brĐsketai sta dexiˆ thc arqik c tou jèshc, ìla ta swmatÐdia ja èprepe na kinhjoÔn tautoqrìnwc proc ta dexi˜, to opoÐo èqei sÐgoura mhdenik pijanìthta.

An kai o parap˜nw tÔpoc apaiteÐ $p > 0$, h d lwsh èqei nìhma ìtan $p = 0$. H diadikasÐa ìpou to $p = 0$ kai ta swmatÐdia metakinoÔntai mìno proc ta arister˜ onom˜zetai pl rwc asÔmmetrh apl diadikasÐa apokleismoÔ.

Epiplèon, ja parajèsoume to asumptwtikì apotèlesma ìtan ta m kai x kai ta dÔo phgaÐnoun sto ˜peiro. QrhsimopoioÔme ton sumbolismì

$$= m=t; \quad c_1 = 1 + 2\sqrt{\ }; \quad c_2 = {}^{1=6}(1 \quad \sqrt{\ })^{2=3}:$$

'Otan 0 p q èqoume

$$\lim_{t!1} P \quad \frac{x_m(t= \ ) \quad c_1 t}{c_2 t^{1=3}} \quad s \ = F_2(s)$$

ompoiìmorfa gia se èna sumpagèc uposÔnolo tou $(0; 1)$.

Sto parap˜nw je¸rhma, h sun˜rthsh $F_2(s)$ pou prokÔptei asumptwtik˜ eÐnai h katanom Tracy-Widom. Oi katanomèc Tracy-Widom eÐnai mia oikogèneia katanom¸n pijanot twn pou perigr˜fhkan apì touc Craig Tracy kai Harold Widom [18, 19], kai faÐnetai ìti dièpoun th mègisth idiotim meg˜lwn tuqaÐwn pin˜kwn.

H suswreutik sun˜rthsh pijanìthtac thc katanom c Tracy-Widom mporeÐ na dojeÐ wc olokl rwma

$$F_2(s) = e^{\ R_1^{s} (x \ s)q^2(x)dx}$$

me q na eÐnai h monadik lÔsh sthn exÐswsh Painlevé II

$$q^{00} = sq + 2q^3; \quad \text{me sunoriak sunj kh } q(s) \quad Ai(s) \text{ ìtan } s ! 1 \quad ;$$

ìpou Ai eÐnai h sun˜rthsh Airy.

Kajolikìthta katanom c Tracy-Widom

Se aut thn enìthta ja suzht soume gia thn kajolikìthta pou èqei diapistwjeÐ prìsfata ìti deÐqnei h katanom Tracy-Widom.

EÐnai gegonìc ìti h JewrÐa TuqaÐwn Pin˜kwn èqei brei ènan ter˜stio arijmì efarmog¸n pou kumaÐnontai apì th statistik fusik mèqri th jewrÐa arijm¸n, ta sunduastik˜ kai ta oloklhrwtik˜ sust mata. MetaxÔ twn prìsfatwn exelÐxewn sth jewrÐa tuqaÐwn pin˜kwn, h melèth thc megalÔterhc idiotim c $_{max}$ meg˜lwn tuqaÐwn pin˜kwn èqei proselkÔsei idiaÐterh prosoq . Oi pr¸tec erwt seic aforoÔsan tic diakum˜nseic tou $_{max}$, pou an koun sto eurÔtero jèma statistik¸n akraÐwn tim¸n.

O biolìgoc Robert May pragmatopoÐhse to 1972 mia fusik efarmog twn statistik¸n tou $_{max}$, pou eÐnai na par˜sqei èna krit rio thc fusik c stajerìthtac se dunamik˜ sust mata ìpwc ta oikosust mata [14]. Je¸rhse ènan plhjusmì N diakrit¸n eid¸n kai eis gage isqurèc allhlepidr˜seic metaxÔ twn eid¸n. Upèjese ìti oi allhlepidr˜seic metaxÔ zeug¸n eid¸n mporoÔn na montelopoihjoÔn apì tuqaÐo pÐnaka, megèjouc N N. 'Ena fusikì er¸thma eÐnai loipìn: poia eÐnai h pijanìthta $P_{stable}(;N)$, ìpou to antiproswpeÔei th dÔnamh twn allhlepidr˜sewn, ìti to sÔsthma paramènei stajerì ìtan energopoihjoÔn oi allhlepidr˜seic; [14] Met˜ apì merikoÔc upologismoÔc, o May apèdwse ìti to sÔsthma jaírameÐnei stajerì, upì thn proôpìjesh ìti oi idiotimèc $_i$ tou tuqaÐou pÐnaka ikanopoioÔn thn anisìthta:

$$_i \quad 1 \quad 0;$$

gia ìla ta $i = 1; :::; N$. Autì eÐnai profan¸c isodÔnamo me th d lwsh ìti h megalÔterh idiotim $_{max} = \max_{1 \; i \; N} \; _i$ ikanopoieÐ thn anisìthta:

$$_{max} \quad \frac{1}{}:$$

Epomènwc h pijanìthta stajerìthtac tou sust matoc sqetÐzetai fusik˜ me thn katanom thc megalÔterhc idiotim c $_{max}$.

H katanom Tracy-Widom dhmiourg jhke gia pr¸th for˜ to 1992, dÔo dekaetÐec argìtera, apì ton Tracy kai ton Widom [18, 19], oi opoÐoi thn parat rhsan melet¸ntac thn Ðdia ènnoia, dhlad tic diakum˜nseic thc megalÔterhc idiotim c $_{max}$ tuqaÐwn pin˜kwn. Argìtera to 1999, oi Baik, Deift, Johansson[2], anak˜luyan ìti h Ðdia statistik katanom perigr˜fei epÐshc tic diakum˜nseic se akoloujÐec anadiatagmènwn akèraiwn arijm¸n - mia entel¸c anex˜rthth majhmatik ènnoia.

SÔntoma h statistik kampÔlh ˜rqise na emfanÐzetai se montèla se ìlh th fusik kai ta majhmatik˜. Aut h ainigmatik kampÔlh fainìtan na eÐnai o sÔnjetoc x˜delfoc thc gnwst c kampÔlhc kamp˜nac, thc Gaussian katanom c, h opoÐa antiproswpeÔei th fusik diakÔmansh twn anex˜rthtwn tuqaÐwn metablht¸n. 'Opwc kai h Gaussian, h katanom Tracy-Widom parousi˜zei kajolikìthta, èna musthri¸dec fainìmeno sto opoÐo diaforetik˜ mikroskopik˜ gegonìta dÐnoun thn Ðdia sullogik sumperifor˜.

Sunart seic puknìthtac pijanìthtac Kanonik c kai Tracy-Widom katanom c

'Otan h katanom Tracy-Widom emfanÐsthke sto prìblhma twn akèraiwn akolouji¸n kai se ˆlla plaÐsia pou den eÐqan kamÐa sqèsh me th jewrÐa tuqaÐwn pinˆkwn, oi ereunhtèc ˆrqisan na yˆqnoun gia to "krummèno n ma pou sundèei ìlec tic ptuqèc mazÐ, ìpwc oi majhmatikoÐ ton 18o kai 19o ai¸na anazhtoÔsan èna je¸rhma pou ja exhgoÔse thn pantaqoÔ paroÔsa kanonik katanom .

To kentrikì oriakì je¸rhma, to opoÐo telikˆ ègine austhrì prin apì perÐpou ènan ai¸na, pistopoieÐ ìti oi fusikèc parathr seic kai ˆllec "mh susqetismènec metablhtèc - pou shmaÐnei ìti opoiad pote apì autèc mporoÔn na allˆxoun qwrÐc na ephreˆsoun tic upìloipec - ja apoteloÔn kampÔlh kampˆnac. AntÐjeta, h kampÔlh Tracy-Widom faÐnetai na prokÔptei apì metablhtèc pou susqetÐzontai èntona, ìpwc h allhlepÐdrash metaxÔ twn eid¸n, oi timèc twn metoq¸n kai oi idiotimèc twn pinˆkwn. O kÔkloc anˆdrashc twn amoibaÐwn apotelesmˆtwn metaxÔ twn susqetismènwn metablht¸n kajistˆ th sullogik sumperiforˆ touc pio perÐplokh apì aut twn mh susqetismènwn metablht¸n.

En¸ oi ereunhtèc èqoun apodeÐxei austhrˆ orismènec kathgorÐec tuqaÐwn pinˆkwn stic opoÐec epikrateÐ genik¸c h katanom Tracy-Widom, èqoun perikukl¸sei ligìtero tic emfanÐseic thc se probl mata katamètrhshc, probl mata tuqaÐwn peripˆtwn, montèla anˆptuxhc kai ˆlla.

Mèqri stigm c, èqoun qarakthristeÐ treic morfèc dianom c Tracy-Widom. EÐnai metasqhmatismènec ekdoqèc h mÐa thc ˆllhc pou perigrˆfoun isqurˆ susqetismèna sust mata me diaforetikoÔc tÔpouc eggenoÔc tuqaiìthtac, dhlad diaforetikoÔc tÔpouc tuqaÐwn pinˆkwn. Sugkekrimèna, oi treic klˆseic tou F (s) shmei¸nontai apì = 1; 2; 4. Wstìso, ja mporoÔsan na upˆrqoun perissìterec apì treic, Ðswc akìmh kai ènac ˆpeiroc arijmìc, tˆxewn.

Kl^seic katanom c Tracy-Widom

Ton teleutaÐo kairì suzhteÐtai ekten¸c to gegonìc ìti o asÔmmetroc qarakt rac thc katanom c mporeÐ na antiproswpeÔei k^poio eÐdoc kajolik c met^bashc f^shc [13]. Sto montèlo tou oikosust matoc tou May, gia par^deigma, to krÐsimo shmeÐo sto $\sqrt{2N}$ qwrÐzei mia stajer f^sh twn asjen¸n suzeugmènwn eid¸n, twn opoÐwn oi plhjusmoÐ mporoÔn na kumaÐnontai memonwmèna qwrÐc na ephre^zontai ta upìloipa, apì mia astaj f^sh twn isqur^ suzeugmènwn eid¸n, ìpou oi diakum^nseic "taxideÔoun isqur^ mèsw tou oikosust matoc kai mporoÔn na to pet^xoun ektìc isorropÐac. Genik^, ta sust mata sthn t^xh kajolikìthtac Tracy-Widom parousi^zoun mÐa f^sh sthn opoÐa ìla ta sustatik^ energoÔn se sunennìhsh, h arister our^, kai mia ^llh f^sh sthn opoÐa ta sustatik^ energoÔn mìna touc, h dexi^ our^.

Aut th stigm , polloÐ fusikoÐ kai majhmatikoÐ erg^zontai ston tomèa thc epidÐwxhc k^poiou kajolikoÔ nìmou pou sundèetai me thn katanom Tracy-Widom. E^n epiteuqjeÐ èna tètoio katìrjwma, ja mporoÔsame na ermhneÔsoume ta makroskopik^ qarakthristika twn susthm^twn me allhlepidr¸nta sustatik^ me polÔ pio fusikì trìpo.

DiadiakasÐec MhdenikoÔ EÔrouc

Se aut n thn enìthta ja melet soume èna ^llo eurèwc gnwstì montèlo Susthm^twn Allhlepidrìntwn SwmatidÐwn. Eis qjh epÐshc arqik^ wc èna aplì par^deigma miac diadikasÐac Markov sto [16]. Onom^zetai DiadikasÐa MhdenikoÔ EÔrouc kai to ìnom^ thc proèrqetai apì to gegonìc ìti ta swmatÐdia allhlepidroÔn mìno me swmatÐdia pou k^jontai ston Ðdio q¸ro.

Montèlo

O q¸roc katˆstashc eÐnai $N^{T^d_L}$. H diadikasÐa kajorÐzetai apì th sunˆrthsh $g : N \to R_+$ me $g(0) = 0$, pou antiproswpeÔei ton rujmì me ton opoÐo èna swmatÐdio feÔgei apì mia jèsh kai mia pijanìthta metˆbashc $p(;)$ sto Z^d. H dunamik thc phgaÐnei wc ex c. Eˆn upˆrqoun k swmatÐdia se mia jèsh x, anexˆrthta apì ton arijmì twn swmatidÐwn se ˆllouc q¸rouc, me rujmì $g(k)p(x; y)$ èna apì ta swmatÐdia sto x metabaÐnei sto y. Me autì ton trìpo ta swmatÐdia allhlepidroÔn mìno me swmatÐdia sthn Ðdia jèsh. Upojètoume epÐshc ìti h g eÐnai austhrˆ jetik sto sÔnolo twn jetik¸n akeraÐwn kai ìti èqei fragmènh diakÔmansh, dhlad :

$$g := \sup_{k \ 0} |g(k+1) \quad g(k)| < 1 :$$

T¸ra, èstw $Z : R_+ \to R_+$ eÐnai h sunˆrthsh diamèrishc me

$$Z( ) = \sum_{k \ 0} \frac{k}{g!(k)}$$

kai èstw $_c$ h aktÐna sÔgklishc thc Z. Sthn teleutaÐa sqèsh eÐnai $g!(k) = \prod_{1 \ j \ k} g(j)$ kai $g!(0) = 1$. Epiplèon, parathr ste ìti h Z eÐnai analutik kai austhrˆ aÔxousa sto $[0; \ _c)$. Upojètoume ìti h $Z( )$ auxˆnei sto 1 ìtan to sugklÐnei sto $_c$:

$$\lim_{\to \ c} Z( ) = 1 :$$

Aut h upìjesh den eÐnai aparaÐthth gia ton orismì thc diadikasÐac, allˆ ja qreiˆzetai gia na apodeÐxoume arketˆ apotelèsmata. Wc ek toÔtou, protimˆtai na sumperilhfjeÐ ston orismì. O genn torac

$$(Lf)( ) = \sum_{x \in T^d_L} \sum_{z \in T^d_L} p^L(z) g((x)) (f(^{x;x+z}) \quad f( ))$$

kajorÐzei mia diadikasÐa Markov sto $N^{T^d_L}$, pou onomˆzetai diadikasÐa mhdenikoÔ eÔrouc me paramètrouc $(g; p)$. EpÐshc ed¸, ìpwc kai se prohgoÔmenh enìthta, h $^{x;y}$ antiproswpeÔei th diamìrfwsh ìpou èna swmatÐdio p dhxe apì to x sto y kai h $p^L( )$ antiproswpeÔei thn pijanìthta metˆbashc pou metakineÐtai sthn arq kai periorÐzetai ston tìmo:

$$p^L(z) := p^L(0; z) = \sum_{y \in T^d_L} p(0; z + yL)$$

gia kˆje d-diˆstato akèraio z.

Se diergasÐec mhdenik c embèleiac kˆje swmatÐdio phdˆ, anexˆrthta apì ta swmatÐdia pou kˆjontai se ˆllec topojesÐec, apì to x sto y me rujmì

$$p^L(y \quad x) g((x)) \frac{1}{(x)} :$$

AnalloÐwto Mètro

Strèfoume t¸ra thn prosoq mac, wc sun jwc, ston qarakthrismì twn a-nalloÐwtwn mètrwn gia th diadikasÐa. Dedomènou ìti h diadikasÐa mhdenikoÔ eÔrouc eÐnai mh upobibˆsimh kai o q¸roc katˆstashc eÐnai peperasmènoc, èqou-me èna monadikì analloÐwto mètro pou upodhl¸noume me $^{N;L}$. Ja anafer-joÔme sta mètra $^{N;L}$ wc kanonikˆ. MporoÔn na upologistoÔn me saf neia, allˆ mporoÔn epÐshc na lhfjoÔn kai me thn dèsmeush twn megˆlwn kanoni-k¸n mètrwn, twn opoÐwn o orismìc akoloujeÐ, epÐ tou sunolikoÔ arijmoÔ twn swmatidÐwn.

Gia kˆje 0 < $_c$, èstw $_{;g}$ = $^L_{;g}$ to megˆlo kanonikì mètro sto $N^{T^d_L}$ me

$$_{;g}\, f \;;\quad (x) = kg = \frac{1}{Z(\ )}\frac{^k}{g!(k)}$$

gia kˆje k 0 kai x 2 T^d_L.

Prìtash. Gia kˆje 0 < $_c$ to mètro $_{;g}$ eÐnai analloÐwto gia th diadikasÐa mhdenikoÔ eÔrouc me tic paramètrouc (g; p). Epiplèon, h su-mplhrwmatik diadikasÐa se sqèsh me opoiod pote apì ta mètra $_{;g}$ eÐnai h diadikasÐa mhdenikoÔ eÔrouc me tic paramètrouc (g; p). Eidikìtera, eˆn to p eÐnai summetrikì, h diadikasÐa eÐnai auto-sumplhrwmatik . [11]

Apìdeixh. H apìdeixh basÐzetai stouc Ðdiouc upologismoÔc pou kˆname gia ta mètra Bernouli kai sthn parakˆtw tautìthta

$$g(k)\frac{^k}{g!(k)}\frac{^j}{g!(j)} = g(j+1)\frac{^{k\ 1}}{g!(k\ \ 1)}\frac{^{j+1}}{g!(j+1)}:$$

EpÐshc, epeid h sunˆrthsh g( ) ja èqei stajer perigraf , gia na diath-r soume aploÔc touc sumbolismoÔc, paraleÐpoume thn exˆrthsh apì to g tou mètrou $_{;g}$ kai to grˆfoume aplˆ me . Kai ètsi,

$$^{N;L}(\ ) = \left(\sum_{x 2 T^d_L} (x) = N \right):$$

T¸ra, sumbolÐzoume me ( ) thn anamenìmeh tim thc metablht c diamìr-fwshc, dhlad thc puknìthtac, kˆtw apì :

$$(\ ) = E\ [\ (x)] = \frac{1}{Z(\ )}\sum_{k\ 0} k\frac{^k}{g!(k)}:$$

To eÔroc thc eÐnai to [0; $_c$), me (0) = 0 kai

$$_c = \lim_{!\ c} (\ )$$

h krÐsimh puknìthta. EpÐshc apì thn parapˆnw exÐswsh eÔkola paÐrnoume ìti:

$$(\ ) = \frac{Z^{0}(\ )}{Z(\ )} = \ @ \ \log Z(\ ):$$

O upologismìc thc pr¸thc parag¸gou thc ( ) deÐqnei ìti eÐnai austhr¸c aÔxousa.

ShmeÐwsh.'Ena fusikì antikeÐmeno endiafèrontoc eÐnai na diereun sou- me th sumperiforˆ aut¸n twn mètrwn sto jermodunamikì ìrio, dhlad ìtan $N; L\ !\ 1$ me tètoio trìpo ¸ste h mèsh puknìthta swmatidÐwn $N=L$ na su- gklÐnei se èna stajerì . Loipìn, sthn upokrÐsimh perÐptwsh, ìtan < $_c$, upˆrqei tètoio ¸ste = ( ) kai h tupik isodunamÐa twn mètrwn gia a- nexˆrthtec tuqaÐec metablhtèc isqÔei kanonikˆ [11]. Dhlad , ta peperasmèna diastasiakˆ perijwriakˆ twn kanonik¸n mètrwn $^{N;L}$ sugklÐnoun sto megˆlo kanonikì mètro pou antistoiqeÐ sto . H isodunamÐa twn mètrwn gia krÐsimec kai uperkrÐsimec puknìthtec, ìtan $_c$, apodeÐqjhke sto [7]. Qrhsimopoi- ¸ntac mejìdouc sqetik c entropÐac, oi suggrafeÐc apodeiknÔoun th sÔgklish twn peperasmènwn diastasiak¸n perijwrÐwn tou $^{N;L}$ sto megˆlo kanonikì mètro sthn krÐsimh katˆstash $_c$. Argìtera sto [1], oi suggrafeÐc èdeixan ìti sto jermodunamikì ìrio oi q¸roi èqoun koin katanom Ðsh me to megˆlo kanonikì mètro se krÐsimh puknìthta, ektìc apì mÐa jèsh pou filoxeneÐ ènan makroskopikˆ megˆlo arijmì swmatidÐwn.

Epiplèon, èqoume suqnˆ parametropoi sei ta analloÐwta mètra apì thn posìthta pou diathreÐte, h opoÐa ed¸ eÐnai h puknìthta twn swmatidÐwn. Gia to lìgo autì, allˆzoume metablhtèc ston orismì twn ametˆblhtwn mètrwn wc ex c. Gia 0, orÐste to mètro wc

$$(\ ) = \ _{(\ )}(\ );$$

ìpou h ( ) orÐzetai wc

$$(\ ) = \begin{cases} \text{antÐstrofh thc } (\ ), & \text{gia } < \ _c \\ \ _c, & \text{gia } \ _c \end{cases}$$

Sto epìmeno l mma faÐnetai ìti h upìjesh pou kˆname gia th sunˆrthsh dia- mèrishc Z egguˆtai ìti to eÔroc thc sunˆrthshc ( ) eÐnai ìlo to R_+. Me autì ton trìpo apokt same mia oikogèneia f : 0g analloÐwtwn mètrwn pou parametropoi jhkan apì thn puknìthta, afoÔ h anamenìmenh tim twn metablht¸n diamìrfwshc (x) kˆtw apì isoÔtai me :

$$E\ [\ (x)] = E_{(\ )}[\ (x)] = \ (\ (\ )) =$$

gia kˆje 0. Akìma, me ènan aplì upologismì blèpoume ìti h sunˆrthsh ( ) eÐnai h anamenìmehnh tim g((0)) kˆtw apì to mètro :

$$(\ ) = E\ [g(\ (0))]:$$

L mma. 'Eqoume shmei¸sei wc $_c$ thn aktÐna sÔgklishc thc sun˜rthshc diamèrishc Z.

$$_c = \lim_{! \;\; c} (\;) = 1 :$$

EpÐshc, gia k˜je 0 < $_c$ gia to mètro , up˜rqei ( ) > 0 tètoio ¸ste

$$E \; [e^{(0)}] < 1 :$$

Ac doÔme t¸ra ti sumbaÐnei sthn perÐptwsh ìpou $_c$ < 1 . Wc apotèlesma, up˜rqei mia krÐsimh puknìthta perib˜llontoc kai ta pleon˜zonta swmatÐdia sumpukn¸nontai se èna mh ektetamèno kl˜sma tou ìgkou. Pio sugkekrimèna, an > $_c$ tìte

$$\lim_{\substack{N;L\,!1 \\ N=L\,!}} {}^{N;L} \left(\max_{x 2 T^d_L} (x) \quad (\qquad _c) L \right) = 1 :$$

En¸, sthn upokritik perÐptwsh, to mègejoc tou megalÔterou stoiqeÐou eÐnai thc t˜xhc log(L) [9]. An jèlame na bg˜loume thn eikìna tou sust matoc sthn uperkrÐsimh f˜sh, ja dianèmame to megalÔtero mèroc twn topojesi¸n sÔmfwna me anex˜rthta antÐgrafa tou $_c$ kai ja susswreÔoume ìlh thn uperbolik m˜za se èna mìno tuqaÐa entopismèno q¸ro.

Ja exet˜soume t¸ra thn perÐptwsh ìpou to g(k) mei¸netai sto k, to opoÐo sth sunèqeia prokaleÐ mia pragmatik èlxh metaxÔ swmatidÐwn. Sugkekrimèna, oi rjumoÐ ˜lmatoc ja eÐnai

$$g(k) = \begin{cases} 0, & \text{an } k = 0 \\ 1 + \frac{b}{k}, & \text{an } k \quad 1 \end{cases}$$

ìpwc prot˜jhke sto [3]. O suggrafèac, epÐshc, parat rhse gia b > 2 ìti $\lim_{! \;\; c} Z(\;)$ kai $_c$ eÐnai kai ta dÔo peperasmèna.

Pio sugkekrimèna, gia b > 2:

$$Z(\;) \; ! \;\; Z(1) = \frac{b}{b \quad 1};$$

$$(\;) \; ! \quad _c = \frac{1}{b \quad 2};$$

ìtan ! 1. Epiplèon, gia b > 3 h deÔterh rop ^{2} thc katanom c $_1$ eÐnai epÐshc peperasmènh, me

$$^2 = \frac{(b \quad 1)^2}{(b \quad 2)^2 (b \quad 3)};$$

kai o arijmìc twn swmatidÐwn ikanopoieÐ to kentrikì oriakì je¸rhma

$$\frac{\sum (x) \quad _c L}{\sqrt{L}} \quad N \;\; (0; 1):$$

'Ena akìmh pr˙gma pou prèpei na shmeiwjeÐ eÐnai ìti h amet˙blhth dianom pou diereun jhke mèqri t¸ra den perièqei plhroforÐec sqetik˙ me th dunamik thc sumpÔknwshc. Mia fusik arqik sunj kh eÐnai na arqÐsoume me swma-tÐdia omoiìmorfa katanemhmèna sthn uperkrÐsimh puknìthta c. Sthn arq ta pleon˙zonta swmatÐdia sumpukn¸nontai se merikèc tuqaÐec jèseic. Mia tètoia topojesÐa pou perièqei poll˙ pleon˙zonta swmatÐdia onom˙zetai perioq sumplègmatoc. Stic upìloipec topojesÐec, pou onom˙zontai mazikèc topojesÐec, h dianom qalar¸nei se $_1$. Me ton auxanìmeno qrìno, ta mega-lÔtera sm nh ja kerdÐsoun swmatÐdia se b˙roc twn mikrìterwn, prokal¸ntac thn exaf˙nish k˙poiwn apì ta sm nh. Telik˙ mìno mÐa om˙da pou perièqei ìla ta pleon˙zonta swmatÐdia epibi¸nei, k˙ti pou eÐnai tupikì gia thn ame-t˙blhth katanom , ìpwc suzht jhke arqÐzontac apì aut n thn enìthta. Ja parathr soume thn parap˙nw eikìna argìtera stic prosomoi¸seic mac.

ProsomoÐwsh

Se aut thn enìthta blèpoume mia apotelesmatik mèjodo gia thn prosomo-Ðwsh susthm˙twn swmatidÐwn me allhlepÐdrash. Se ìlec tic prohgoÔmenec enìthtec orÐzoume th qronik exèlixh twn susthm˙twn mac na eÐnai suneq c kai autì eÐnai k˙ti aparaÐthto gia na uposthrÐxoume ˙llec ènnoiec ìpwc ta ìria klim˙kwshc. Wstìso, ìpwc gnwrÐzete, se ènan upologist , h exèlixh prèpei na sumbaÐnei se diakritoÔc qrìnouc.

OrÐzoume mia alusÐda Markov na eÐnai mia diadikasÐa Markov orismènh se èna metr simo sÔnolo kai na qarakthrÐzetai apo touc rujmoÔc met˙bashc $c(\;;\;{}^0) \quad 0$. Epiplèon, shmei¸noume me

$$W \;:= \inf f t \quad 0 : \;_t \neq \; g$$

ton qrìno paramon c se mia kat˙stash , kai me

$$c \;:= \sum_{{}^0 \neq} c(\;;\;{}^0):$$

touc sunolikoÔc rujmoÔc exìdou apo thn . Profan¸c an $c = 0$, tìte $W = 1$. Akìma, isqÔei ìti an $c > 0$, tìte

$$W \quad \mathrm{Exp}(c\;);$$

ìpou me $\mathrm{Exp}(\;)$ shmei¸noume thn ekjetik katanom me par˙metro .

'Oso prosomoi¸noume èna sÔsthma swmatidÐwn me allhlepÐdrash se ènan tìmo T_L^d, up˙rqei p˙nta ènac meg˙loc brìqoc, ston opoÐo k˙je epan˙lhyh antiproswpeÔei mia met˙bash, dhlad to ˙lma enìc swmatidÐou. H epilog tou swmatidÐou sundèetai sten˙ me to qrìno paramon c sthn kat˙stash tou. Shmei¸ste ìti oi metabatikoÐ rujmoÐ gia th diadikasÐa mhdenikoÔ eÔrouc eÐnai:

$$c(\;;\;{}^{x;y}) = g(\;(x))\,p(y \quad x)\,\frac{1}{(x)}:$$

Ta b mata prosomoÐwshc thc diadikasÐac mhdenikoÔ eÔrouc ekfr˘zontai apì ton parak˘tw algìrijmo.

```
rates    sumf g( (x)) gia k˘je x sto T^d_L g
time    0
while true do
  epèlexe èna x me pijanìthta g( (x))=rates
  epèlexe èna y me pijanìthta p(y   x)
   (x)     (x)   1
   (y)     (y) + 1
  time    time + Exp(rates)
  rates    g( (x)) + g( (y))   g( (x) + 1)   g( (y)   1)
end while
```

PisteÔoume ìti ìla ta b mata eÐnai arket˘ apl˘. Shmei¸noume bèbaia pwc isqÔei ìti

$$\min(\mathrm{Exp}(\ _1); ::: ; \mathrm{Exp}(\ _n)) \quad \mathrm{Exp}(\sum_{i=1}^{n} \ _i):$$

To parap˘nw aitiologeÐ tic aux seic thc metablht c times.

Peiramatik˘ Apotelèsmata

Anex˘rthtoi TuqaÐoi PerÐpatoi

To stoqastikì montèlo twn anex˘rthtwn tuqaÐwn perip˘twn prosdiorÐsthke sthn pr¸th enìthta. Epiplèon, apodeÐqjhke ìti gia èna sÔsthma mh diaqwrÐsimwn swmatidÐwn pou akoloujoÔn tuqaÐouc perip˘touc, up˘rqei mia monadik oikogèneia apo analloÐwta mètra, parametropoi simh apì thn puknìthta twn swmatidÐwn, pou onom˘zontai Poisson mètra.

Arqik˘, ac epibebai¸soume to apotèlesma thc analloÐwthc katanom c. JewroÔme èna monodi˘stato tìmo me 1000 q¸rouc. Sth sunèqeia, katanèmoume arqik˘ ta swmatÐdia sÔmfwna me to mètro Poisson kai parathroÔme th dianom se metagenèsterec qronikèc stigmèc t˘xhc N kai N^2. Sta epìmena sq mata, blèpoume, gia diaforetikèc puknìthtec swmatidÐwn, ìti autì to apotèlesma, pr˘gmati, isqÔei.

Katanom se qrìno t ston T^{1}_{1000} me   = 5

Katanom se qrìno t ston T^{1}_{1000} me   = 10

Epiplèon, jèlame na apeikonÐsoume thn exèlixh enìc sust matoc anexˆrth-twn tuqaÐwn peripˆtwn se dÔo diastˆseic. Gia na to epitÔqoume autì, p rame mia eikìna pou antiproswpeÔei ton disdi˜stato tìmo kai dhmiourg same mia logarijmik klÐmaka qr¸matoc gia ton arijmì twn swmatidÐwn se k˜je jèsh. SÔmfwna me thn kanonikopoihmènh logarijmik tim tou arijmoÔ twn swmatidÐwn, to qr¸ma ex˜getai sto akìloujo f˜sma:

ˆspro ! kÐtrino! kìkkino ! maÔro.

Stic eikìnec pou parousi˜zontai parak˜tw, blèpoume thn exèlixh tou sust matoc, me puknìthta swmatidÐwn 10, met˜ apì k˜poio qronikì di˜sthma, me ta swmatÐdia na brÐskontai arqik˜ sto kèntro tou tìmou. Sthn pr¸th eikìna, h stoiqei¸dhc pijanìthta met˜bashc f p(x; y) : (x; y) 2 T^2_{100} g eÐnai summetrik , en¸ sto deÔtero asÔmmetrh me:

p(1; 0) = 0 :3 (dexi˜), p(1; 0) = 0 :1 (arister˜),

p(0; 1) = 0 :5 (p˜nw), p(0; 1) = 0 :1 (k˜tw).

SummetrikoÐ TuqaÐoi PerÐpatoi sto T^2_{100}

AsÔmmetroi TuqaÐoi PerÐpatoi sto Z^2_{100} se qrìno $t_1 > 0$

AsÔmmetroi TuqaÐoi PerÐpatoi sto Z^2_{100} se qrìno $t_2 > t_1$

Apl DiadikasÐa ApokleismoÔ

Sth sunèqeia, ja melet soume thn apl diadikasÐa apokleismoÔ. 'Opwc eÐdame se prohgoÔmenh enìthta, gia thn apl diadikasÐa apokleismoÔ, upˆrqei mia monadik oikogèneia apo analloÐwta mètra, parametropoi simh apì thn puknìthta twn swmatidÐwn, pou onomˆzontai Bernoulli mètra.

Argìtera s' aut n thn enìthta, parousiˆsame thn asÔmmetrh apl diadikasÐa apokleismoÔ me bhmatik arqik katˆstash. Se autì to shmeÐo ja ektelèsoume peirˆmata sqetikˆ me thn sumperiforˆ thc apìstashc jX_1j pou to perijwriakì swmatÐdio èqei kalÔyei se èna dosmèno qrìno.

Pr¸ta, ja diapist¸soume thn exˆrthsh thc mèshc tim c tou $jX_1(t)j$ apì ton qrìno t, tìso gia entel¸c asÔmmetrec ìso kai gia mh entel¸c asummetrikèc peript¸seic. Upojètoume ìti

$$E(jX_1(t)j) \quad ct \;;$$

pou ekfrˆzetai grammikˆ

$$\log E(jX_1(t)j) \qquad \log t + \log c:$$

Gia thn entel¸c asÔmmetrh perÐptwsh br kame

$$E(jX_1(t)j) \quad t:$$

T¸ra, jumhjeÐte ton sumbolismì $= q \quad p$, ìpou p eÐnai h pijanìthta èna swmatÐdio na phdˆei mÐa jèsh sta dexiˆ kai q pou phdˆei mia jèsh sta aristerˆ. En¸ sthn ˆkrwc asÔmmetrh perÐptwsh eÐnai $= 1$, jewroÔme epÐshc thn asÔmmetrh perÐptwsh ìpou $= \frac{3}{4} \quad \frac{1}{4} = \frac{1}{2}$. To apotèlesma sthn perÐptwsh aut tan

$$E(jX_1(t)j) \quad \frac{t}{2} = \quad t:$$

Tèloc, ja mporoÔse kaneÐc na to pei ìti

$$E(jX_1(t= \;)j) \quad t$$

gia kˆje $\quad 2\ (0;1]$.

Sth sunèqeia, prèpei na kajorÐsoume th seirˆ twn diakumˆnsewn gÔrw apì th mèsh tim ìso auxˆnetai o qrìnoc. Akolouj¸ntac thn Ðdia strathgik ìpwc prohgoumènwc, diapist¸same ìti gia thn entel¸c asÔmmetrh perÐptwsh

$$jX_1(t)j \quad E(jX_1(t)j) \quad O\ (\sqrt{t}):$$

T¸ra ìson aforˆ tic diakumˆnseic thc mh entel¸c asÔmmetrhc èkdoshc, diapist¸same ìti

$$jX_1(t)j \quad E(jX_1(t)j) \quad O\ (t^{0:6}):$$

Kai sth sunèqeia, exetˆzontac thn akìloujh grafik parˆstash thc katanom c thc metakinhmènhc kai kanonikopoihmènhc tim c tou $jX_1(t)j$ gia th mh entel¸c asÔmmetrh perÐptwsh kai ton akìloujo pÐnaka parathroÔme mia exairetik omoiìthta me thn katanom Tracy-Widom.

Katanom tou $jX_1(t)j$ me $= 0:5$

Mèsh tim	Diaspor^	TrÐth Rop	Pleìnasma Tètarthc Rop c
-0.93205	0.241756	0.23716	0.0714276

Qarakthristik^ thc katanom c

DiadikasÐa MhdenikoÔ EÔrouc

Se autì to tm ma ja ektelèsoume prosomoi¸seic se diadikasÐec mhdenikoÔ eÔrouc, lamb^nontac upìyh to montèlo touEvans [3], ìpou

$$g(k) = \begin{cases} 0, & \text{an } k = 0 \\ 1 + \frac{b}{k}, & \text{an } k \quad 1 \end{cases}$$

Epiplèon, ja exet^soume mìno tic uperkritikèc peript¸seic me puknìthta $>$ $_c$ kai $b > 2$ gia na diasfalÐsoume ìti $_c < 1$.

H akìloujh eikìna antiproswpeÔei ènan disdi^stato tìmo exoplismèno me thn kanonikopoihmènh logarijmik klÐmaka qr¸matoc pou perigr^fthke par^nw. Prokeimènou na to par^goume, èqoume af sei mia diadikasÐa mhdenikoÔ eÔrouc, me swmatÐdia omoiìmorfa arqik˘ katanemhmèna, na exelÐsetai gia èna shmantikì qronikì di˘sthma t. Sth sunèqeia, k˘je topojesÐa sthn eikìna a-ntiproswpeÔei to mègisto arijmì swmatidÐwn pou èqoun filoxenhjeÐ se autìn ton q¸ro gia k˘poio qronikì di˘sthma. Me autèc tic plhroforÐec mporoÔme na exag˘goume sqedìn ìlh thn exèlixh tou sust matoc apì to qrìno 0 me ton akìloujo trìpo.

'Opwc perigr˘yame sthn enìthta twn diadikasi¸n mhdenikoÔ eÔrouc, xe-kin¸ntac apì thn arqik kat˘stash me omoiìmorfa katanemhmèna swmatÐdia sthn uperkrÐsimh puknìthta, ta pleon˘zonta swmatÐdia sumpukn¸nontai se merikèc tuqaÐec jèseic, pou onom˘zontai jèseic sumpuknwm˘twn. Me ton au-xanìmeno qrìno, ta megalÔtera sm nh ja kerdÐsoun swmatÐdia se b˘roc twn mikrìterwn, prokal¸ntac thn exaf˘nish orismènwn sumpuknwm˘twn. Me au-tìn ton trìpo, an epilèxete mia puknìthta katwflÐou $_{th}$ > $_{c}$ kai filtr˘rete thn eikìna, tìte ja èqete tic topojesÐec sumpuknwm˘twn gia k˘poia aujaÐreth ¸ra. Kai an aux sete aut thn puknìthta, tìte ja èqete ligìterec topojesÐec sumpuknwm˘twn pou antistoiqoÔn se k˘poia metèpeita stigm .

DiadikasÐsa MhdenikoÔ EÔrouc (x) = max $_{0 \; t^0 \; t \; t^0}$(x)

T¸ra, ja prospaj soume na kajorÐsoume thn t˘xh tou qrìnou pou apaite-Ðtai gia na epiteuqjeÐ isorropÐa sto sÔsthma. Ja ergastoÔme se mÐa di˘stash. JumhjeÐte ed¸ ìti h kat˘stash isorropÐac perièqei èna mìno sump knwma kai ìloi oi upìloipoi q¸roi katanemhmènoi sÔmfwna me to $_{c}$. Exet˘same tic akìloujec treic arqikèc katanomèc:

1. (x) = ; x 2 T_L

2. (0) = L 2 $_c$(L 1) kai (x) = 2 $_c$; x 2 T_L=f 0g

3. (0) = L 4 $_c$L=5,
(x) = 4 $_c$; x 2 [2L=5; 3L=5), alli¸c (x) = 0

H teleutaÐa perÐptwsh emfanÐzetai wc ex c.

Par^deigma me L = 1000, p = 1 , p_c = 0:1

An kai o pragmatikìc qrìnoc gia thn epÐteuxh isorropÐac apì autèc tic katast^seic eÐnai diaforetikìc, h t^xh tou qrìnou eÐnai h Ðdia. AfoÔ xekin same prosomoi¸seic gia dÔo diaforetikèc peript¸seic stoiqei¸douc pijanìthtac ^lmatoc, br kame ta akìlouja apotelèsmata. Se mia entel¸c asÔmmetrh diadikasÐa mhdenikoÔ eÔrouc, dhlad p(1e) = 1 , èqoume ìti

$$T_{eq} = O(L^2):$$

'Otan se mia summetrik diadikasÐa mhdenikoÔ eÔrouc, dhlap(m1e) = p(1) = 1=2, èqoume ìti

$$T_{eq} = O(L^3):$$

Perimèname o qrìnoc sthn summetrik perÐptwsh na eÐnai uyhlìterhc t^xhc kaj¸c ta swmatÐdia diaqèontai qwrÐc sugkekrimènh kateÔjunsh.

Sth sunèqeia, ja qrhsimopoi soume thn trÐth arqik katanom pou perigr^yame parap^nw kai ja melet soume th dunamik enìc swmatidÐou me etikèta, tou opoÐou h jèsh ja parakoloujeÐtai, se antÐjesh me ta ^lla mh diakrÐsima swmatÐdia. Jewr ste tic akìloujec treic kathgorÐec enìc swmatidÐou me etikèta. E^n èna swmatÐdio prìkeitai na metaphd sei apì thn topojesÐa ìpou brÐsketai to etiketopoihmèno swmatÐdio, Pr¸th kl^sh: ìte autì to swmatÐdio ja eÐnai p^nta autì me etikèta,

TuqaÐa: ìte autì to swmatÐdio ja eÐnai autì me etikèta me pijanìthta 1= (x),

DeÔterh kl^sh: ìte autì to swmatÐdio ja eÐnai autì me etikèta, an eÐnai to teleutaÐo ston q¸ro.

KaloÔmaste t¸ra na prosdiorÐsoume thn ex^rthsh thc apìstashc tou swmatidÐou me etikèta apì thn arqik tou jèsh apì to qrìno t. To etiketopoihmèno swmatÐdio ja brÐsketai arqik^ sthn topojesÐa $x = L=2$. EpÐshc, to klimak¸noume ton q¸ro me L kai ton qrìno me L^2 [15]. Oi prosomoi¸seic mac èdwsan:

Pr¸th kl^sh: $X_{tag}(tL^2)=L = O(1)$,

TuqaÐa: $X_{tag}(tL^2)=L = O(\sqrt{t})$,

DeÔterh kl^sh: $X_{tag}(tL^2)=L = O(\sqrt{t})$.

Upoyi^zomai ìti ta swmatÐdia me etikèta pr¸thc t^xhc den faÐnetai na exart¸ntai apì to qrìno lìgw thc t^shc touc na ft^noun gr gora ston sumpuknwma.

EpÐlogoc

Ta teleutaÐa qrìnia èqei prokÔyei èntona h an^gkh thc melèthc twn Susthm^twn Allhlepidr¸ntwn SwmatidÐwn, exaitÐac tou eurèoc f^smatoc efarmog¸n se fusik^ probl mata. Sthn paroÔsa ergasÐa, xekin same orÐzontac thn aploÔsterh morf enìc sust matoc swmatidÐwn, dhlad touc anex^rthtouc tuqaÐouc perip^touc, kai sth sunèqeia arqÐsame na jètoume erwt mata sqetik^ me tic upokeÐmenec sumperiforèc. Autèc oi erwt seic peril^mbanan thn Ôparxh amet^blhtwn katanom¸n sto sÔsthma, thn exagwg tou makroskopikoÔ profÐl tou sust matoc se dedomènh qronik kai qwrik klÐmaka, thn oriojèthsh aut¸n twn klim^kwn klp. Met^ apì autì, rje h ¸ra na strèyoume thn prosoq mac se lÐgo pio sÔnjeta sust mata allhlepidr¸ntwn swmatidÐwn, ìpwc h apl diadikasÐa apokleismoÔ kai h diadikasÐa mhdenikoÔ eÔrouc. Aut^ ta montèla mazÐ me pollèc dhmofileÐc diakum^nseic touc èqoun dh melethjeÐ eurèwc apì touc ereunhtèc. Se mÐa apì tic diakum^nseic thc apl c diadikasÐac apokleismoÔ emfanÐzetai h katanom Tracy-Widom. Me thn eukairÐa aut , xekin same mia suz thsh sqetik^ me thn kajolikìthta pou aut h sugkekrimènh dianom faÐnetai na ekdhl¸nei teleutaÐa. Epiplèon melet same tic diadikasÐec mhdenikoÔ eÔrouc, katal xame sthn analloÐwth katanom touc kai eÐdame merikèc idiìthtec se uperkrÐsimec puknìthtec.

Wc mellontikèc grammèc ergasÐac sto pedÐo, proteÐnoume na apokthjeÐ mia bajÔterh katanìhsh twn idiot twn thc katanom c Tracy-Widom, kaj¸c faÐnetai na diadramatÐzei kentrikì rìlo se ènnoiec me allhlepidr¸nta sustatik^.

Epiplèon, ja tan polÔ apodotik h efarmog teqnik¸n meĐwshc diasporˆc se prosomoi¸seic susthmˆtwn allhlepidrìntwn swmatidĐwn.

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