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Φυσικές και Αποδοτικές Δυναμικές σε Παίγνια Διαμόρφωσης Άποψης μέσω τεχνικών Κυρτής Βελτιστοποίησης

Διδακτορική Διατριβή

του

Ευστράτιου Παντελεήμονος Χ. Σκουλάκη

Αθήνα, Σεπτέμβριος 2019

Εθνικο Μετσοβιο Πολυτεχνειο
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Περίληψη

Η παρούσα διδακτορική διατριβή μελετά παίγνια, δυναμικά συστήματα και υπολογιστικά προβλήματα που σχετίζονται με τη διαμόρφωση άποψης. Η εργασία αυτή κινείται σε τρείς βασικούς άξονες. Ο πρώτος άξονας αφορά στη μελέτη των ιδιοτήτων σύγκλισης αλγόριθμων επιλογής στρατηγικών σε παίγνια διαμόρφωσης άποψης που εξελίσσονται στο χρόνο. Εξετάζονται εκτενώς οι ιδιότητες σύγκλισης σε ισορροπία Nash όταν η ανανέωση των απόψεων (στρατηγικών) γίνεται βάσει αλγορίθμων best response και no-regret ακόμα και σε περιπτώσεις που οι παίκτες έχουν μερική γνώση των απόψεων (στρατηγικών) των άλλων παικτών. Ο δεύτερος άξονας αφορά στην επέκταση των άνω φραγμάτων για το Τίμημα της Αναρχίας σε παίγνια διαμόρφωσης άποψης όταν οι απόψεις κάποιων παικτών μπορεί να δουν απωθητικά για τις απόψεις άλλων παικτών. Αποδεικνύεται ότι το Τίμημα της Αναρχίας φράσσεται από μία καθολική σταθερά που δεν εξαρτάται από τον αριθμό των παικτών. Στον τρίτο άξονα της εργασίας, εξετάζεται μία δυναμική εκδοχή του προβλήματος k-median στην οποία οι θέσεις των πελατών βρίσκονται στην ευθεία και εξελίσσονται στον χρόνο. Για το πρόβλημα αυτό παρουσιάζεται ένας αλγόριθμος πολυωνυμικού χρόνου ο οποίος στηρίζεται στην επίλυση ενός κατάλληλου γραμμικού προγράμματος.

Λέξεις κλειδιά: Παίγνια Διαμόρφωσης Άποψης, Αλγοριθμική Θεωρία Παιγνίων, Κυρρτή Βελτιστοποίηση

Abstract

This thesis studies issues related to problems arising in *opinion dynamics* and *opinion formation games*. The way people form their opinions can be modelled as a *no-cooperative game* where each selfish agent strategically selects her opinion so as to minimize her individual disagreement cost. When such a game is repeatedly played over time, agents repeatedly update their opinions (according to the opinions of the other) leading to a dynamics of the opinions.

We examine extensions of the well known opinion dynamics Friendkin Johnsen model and Hegselmann Krause model. Our variants are motivated by natural social phenomena, such as limited information exchange, presence of social structure and influence by global trends, that were not captured by the original models. In the considered settings the convergence properties of the original models are seriously under question. Through the use of ideas and techniques develloped in the context of Convex optimization, we are able to analyze the dynamic behavior of the opinions and to study the quality of equilibrium points in terms of social disagreement cost.

Keywords: Opinion Formation Games, Algorithmic Game Theory, Convex Optimization

Ευχαριστίες

Η παρούσα διδακτορική διατριβή δεν θα μπορούσε να είχε πραγματοποιηθεί χωρίς την συμβολή του επιβλέποντα μου Δημήτρη Φωτάκη. Τον ευχαριστώ από καρδιάς για όλο το ενδιαφέρον, την υπομονή και την καθοδήγηση του. Ευχαριστώ θερμά τον ακαδημαϊκό μου παππού και θείο Στάθη Ζάχο και Άρη Παγουρτζή αντίστοιχα για την συμπαράσταση και την βοήθεια τους όποτε και την χρειάστηκα. Πολλά ευχαριστώ στα ακαδημαϊκά μου αδέρφια και ξαδέρφια Αντώνη Αντώνόπουλο, Αγγέλα Χαλκή, Γιάννη Παπαιωάννου, Παναγιώτη Πατσιλινάκο, Βασίλη Κοντονή, Τζέλα Μαθιουδάκη, Λουκά Κάβουρα, Θανάση Λιανέα, Ναταλία Κοτσανή, Κατερίνα Νικολιδάκη, Σωτήρη Δήμου για την αγάπη, την στήριξη και για όλες τις ομόρφες και δύσκολες στιγμές που περάσαμε παρέα στο θρυλικό Corelab. Θα ήθελα επίσης να ευχαριστήσω όλη τα μέλη της επταμελούς μου επιτροπής καθώς και το Ίδρυμα Ωνάση για την χρηματοδότηση της διδακτορικής μου διατριβής. Τέλος ευχαριστώ την οικογένεια μου (τους γονείς μου, τα αδέρφια μου και τους φίλους μου) που με προσέχουν τα τελευταία 29 και κάτι χρόνια..

Στρατής Σκουλάκης
Αθήνα, Σεπτέμβριος 2019

Contents

Chapter 1

Extended Abstract in greek

Η μελέτη του τρόπου με τον οποίο οι άνθρωποι σχηματίζουν απόψεις έχει μακρά ιστορία [92] . Η διαμόρφωση απόψεων είναι μια *δυναμική διαδικασία* στην οποία κοινωνικά συνδεδεμένοι άνθρωποι ανταλάσσουν πληροφορίες και αυτό οδηγεί στην αλλαγή των απόψεων τους στην πάροδο του χρόνου. Σήμερα, η έλευση του διαδικτύου και των μέσων κοινωνικής δικτύωσης καθιστά αυτή τη μελέτη ακόμα πιο σημαντική. Η κατανόηση των δυναμικών διαμόρφωσης άποψης βρίσκει τεράστιες πρακτικές εφαρμογές στην πρόβλεψη εκλογικών αποτελεσμάτων, στη διαφήμιση κ.λ.π. Στην προσπάθεια συστηματοποίησης αυτής της μελέτης, τα τελευταία χρόνια έχουν προταθεί διάφορα μαθηματικά μοντέλα για την διαμόρφωση άποψης [60, 79, 89, 59] .

Η κοινή παραδοχή των περισσότερων μοντέλων, η οποία χρονολογείται από το DeGroot [60], είναι ότι οι απόψεις εξελίσσονται ως ένα *δυναμικό σύστημα επαλαμβανόμενου μέσου όρου*. Πιο συγκεκριμένα οι κοινωνικές οντότητες μοντελοποιούνται ως πράκτορες που σε κάθε βήμα ανανεώνουν τις απόψεις στο μέσο όρο των απόψεων του κοινωνικού τους κύκλου. Αρχικά ο κάθε πράκτορας έχει μια τιμή που αντιπροσωπεύει την αρχική του άποψη. Σε κάθε γύρο, όλοι οι πράκτορες υιοθετούν ως νέα άποψη ένα κυρτό συνδυασμό των απόψεων των άλλων πρακτόρων στο προηγούμενου γύρο. Με αυτό τον τρόπο δημιουργείται μία *δυναμική* των απόψεων στο χρόνο. Οι συντελεστές αυτού του κυρτού συνδυασμού μπορεί να διαφέρουν από πράκτορα σε πράκτορα και στην πραγματικότητα μπορεί να αλλάζουν με την πάροδο του χρόνου.

Συστήματα επαλαμβανόμενου μέσου όρου

1: n πράκτορες.
2: $x_i(0) \in [0, 1]$, η αρχική άποψη του πράκτορα i.
3: Στο γύρο $t \geq 1$, κάθε πράκτορας i ανανεώνει την άποψη του:

$$x_i(t) = \sum_{j=1}^{n} p_{ij}(t) x_j(t-1)$$

όπου $p_{ij}(t) \geq 0$ και $\sum_{j=1}^{n} p_{ij}(t) = 1$

Ο ακριβής ορισμός των συντελεστών $p_{ij}(t)$ γίνεται στο κάθε συγκεκριμένο μοντέλο. Για παράδειγμα στο μοντέλο DeGroot οι συντελεστές είναι σταθεροί και αμετάβλητοι στο χρόνο ($p_{ij}(t) = p_{ij}$). Ενώ στο μοντέλο Hegselmann Krause ο κάθε πράκτορας βρίσκει τους πράκτορες με άποψη σε απόσταση το πολύ 1 από την δική του και υιοθετεί ως νέα άποψη το μέσο όρο αυτών των απόψεων.

Σε πρώτη ματιά τέτοια δυναμικά συστήματα ίσως μοίαζουν κάπως απλοϊκά για την περιγράφη περίπλοκων φυσικών διαδιασιών όπως είναι η διαμόρφωση άποψης. Αν και δεν είναι εφανές σε πρώτο χρόνο, τέτοια δυναμικά συστήματα έχουν τεράστια εκφραστική δύναμη.

Αρχικά πειραματικές μελέτες σε μικρές κοινότητες στην Ινδία έχουν επαληθεύσει την προγνωστική δύναμη τέτοιων μοντέλων στον τρόπο σχηματισμού των άποψεων [7]. Επίσης τέτοιου είδους δυναμικά συστήματα παρουσιάζουν μεγάλη επιτυχία στην μοντελοποίηση διάφορων φυσικών διαδικασίων που ξεφεύγουν από τα στενά όρια της διαμόρφωσης άποψης. Μερικές πολύ ενδιαφέρουσες εφαρμογές αφορούν στην μοντελεποίηση της συμπεριφοράς των ζώων. Η δημιουργία σμηνών από πουλιά [36, 124], ο συντονισμός κίνησης κοπαδιών απο ψάρια [118, 121] και ο συντονισμός των σημάτων φωτός των πυγαλαμπίδών [111] μπορούν να περιγραφούν με αρκετά ακριβή τρόπο από δυναμικά συστήματα επαλαμβανόμενου μέσου όρου, παρόμοια με αυτά που περιγράφουν την διαμόρφωση των απόψεων. Άλλες ενδιαφέρουσες εφαρμογές τέτοιων συστημάτων αφορούν στον συντονισμό δικτύων αισθητήρων, στην εξέλιξη κυτταρικών πληθυσμών και στον συντονισμό σημάτων βηματοδότη καρδιάς [34].

Εκτός από τις τεράστιες εφαρμογές των συστημάτων επαλαμβανόμενου μέσου όρου στην μοντελεποίηση φυσικών διαδικασιών, τέτοια συστήματα βρίσκουν εφαρμογή και στό χώρο της Θεωρητικής Πληροφορικής. Σύγχρονες επιστημονικές εργασίες δέιχνουν πως τέτοια συστήματα μπορούν να λύσουν υπολογιστικά προβλήματα! Για παράδειγμα, στην ερευνητική εργασία [12] σχεδιάζεται ένα πολύ κομψό δυναμικό σύστημα επαλαμβανόμενου μέσου όρου το οποίο λύνει το πρόβλημα της ανίχνευσης κοινότητας σε γραφήματα. Επίσης στην εργασία [34] αποδεικνύεται πως τέτοια συστήματα μπορούν να είναι ακόμη και Turing complete! Άλλες ενδιαφέρουσες *αλγοριθμικές* εφαρμογές τέτοιων συστημάτων μπορούν να βρεθούν στις εργασίες [95, 97].

Λόγω των παραπάνω εφαρμογών, η μελέτη της δυναμικής των μοντέλων διαμόρφωσης άποψης έχει προσελκύσει έντονα το επιστημονικό ενδιαφέρον. Αυτή η ερευνητική γραμμή προσπαθεί να κατανοήσει την παρακάτω ερώτηση:

Πότε τα εν λόγω συστήματα σύγκλινουν σε σταθερά σημεία;

Δυστυχώς η μεγάλη εκφραστική δύναμη των συστημάτων επαλαμβανόμενου μέσου όρου, επιφέρει ως αποτέλεσμα την δυσκολία στην ανάλυσης τους. Μικρές παραλλαγές των μοντέλων οδηγούν σε εντελώς διαφορετικές δυναμικές συμπεριφορές. Αν και υπάρχουν αποτελέσματα που χαρακτηρίζουν ιδιότητες σύγκλισης γενικών κλάσεων τέτοιων συστημάτων [91, 105, 35] δεν υπάρχει μια ενοποιημένη θεωρία που να περιγράφει τη συμπεριφορά τους. Στην πραγματικότητα κάθε μοντέλο αναλύεται με ξεχωριστό τρόπο και οι ιδέες και τεχνικές διαφέρουν σημαντικά.

Η παρούσα διδακτορική διατριβή ασχολείται με την μελέτη ιδιοτήτων σύγκλισης γενικεύσεων των μοντέλων Friedkin Johnsen και Hegselmann Krause. Τα μοντέλα αυτά αποτελούν από τα σημαντικότερα και πιο εκτενώς μελετημένα μοντέλα για την διαμόρφωση άποψης. Στο μεγαλύτερο μέρος της έρευνας που πραγματοποιήθηκε στα πλαίσια αυτής της διδακτορικής διατριβής, χρησιμοποιήσαμε εργαλεία και τεχνικές που έχουν αναπτυχθεί στο χώρο της Κυρτής Βελτιστοποίησης για την απόδειξη σύγκλισης των γενικεύσεων των παραπάνω μοντέλων. Η γενικεύσεις που μελετήσαμε προέρχονται από διαισθητικές παρατηρήσεις στον τρόπο που διαμορφώνονται οι απόψεις σε μεγάλα κοινωνικά δίκτυα και παρουσιάζονται στο τέλος του κεφαλαίου.

Το μοντέλο Friendkin Johnsen και Πάιγνια Διαμόρφωσης Άποψης

Ένα από τα σημαντικότερα μοντέλα διαμόρφωσης άποψης, προτάθηκε από τους Friedkin και Johnsen το 1990 [79]. Το μοντέλο Friedkin Johnsen προτάθηκε αρχικά σαν μια παραλλαγή του μοντελου DeGroot για να εξηγήσει το γεγονός ότι ομοφωνία στις απόψεις επιτυγχάνεται σπάνια.

Σύμφωνα με το μοντέλο, κάθε πράκτορας i εκφράζει μία δημόσια άποψη $x_i \in [0, 1]$, ενώ παράλληλα έχει μια εσωτερική άποψη $s_i \in [0, 1]$ η οποία είναι σταθερή και αμετάβλητη στην πάροδο του χρόνου. Το μοντέλο επίσης υποθέτει την ύπαρξη ενός γραφήματος $G(V, E, w)$ το οποία αναπαριστά τις κοινωνικές σχέσεις μεταξύ των πρακτόρων. Το σύνολο των κόμβων V αντιπροσωπεύει τους πράκτορες και το σύνολο των ακμών E τις κοινωνικές τους σχέσεις. Το βάρος w_{ij} μιας ακμής $(i,j) \in E$ είναι πάντα θετικό, $w_{ij} \geq 0$, και αντιπροσωπεύει την επιρροή που ασκεί ο πράκτορας j στον πράκτορα i. Επίσης ο κάθε πράκτορας i έχει ενα θετικό βάρος $w_i > 0$ που αποτυπώνει την επιμονή του πράκτορα στην εσωτερική του άποψη. Αρχικά, όλοι οι κόμβοι ξεκινούν με κάποιες αρχικές δημόσιες απόψεις $x_i(0)$ και σε κάθε γύρο t, ανανεώνουν την δημόσια άποψη τους $x_i(t)$ στο σταθμισμένο μέσο όρο των δημόσιων απόψεων των γειτόνων τους και της εσωτερικής τους άποψης.

Το μοντέλο Friedkin Johnsen

1: Ένα γράφημα με βάρη, $G(V,E,w)$, $|V| = n$.
2: $s_i \in [0,1]$, η εσωτερική άποψη του πράκτορα i.
3: $x_i(0) \in [0,1]$, η αρχική δημόσια άποψη του πράκτορα i.
4: Στο γύρο $t \geq 1$ κάθε πράκτορας i ανανεώνει την δημόσια άποψη του:

$$x_i(t) = \frac{\sum_{j=1}^{n} w_{ij} x_j(t-1) + w_i s_i}{\sum_{j=1}^{n} w_{ij} + w_i}$$

Το μοντέλο Friedkin Johnsen έχει ένα πολύ απλό κανόνα ανανέωσης των απόψεων που το καθιστά εύλογο ως μοντέλο, ενώ οι βασικές του παραδοχές ευθυγραμμίζονται με εμπειρικά αποτελέσματα του τρόπου σχηματισμού των απόψεων [2, 102, 7]. Το μοντέλο Friedkin Johnsen έχει μελετηθεί εκτενώς και έχει αποδειχθεί πως έχει ένα μοναδικό σταθερό σημείο $x^* \in [0,1]^n$ στο οποίο συγκλίνει με *γραμμικό ρυθμό* ανεξαρτήτως των αρχικών απόψεων [81].

Το 2011 οι Bindel, Kleinberg και Oren εισήγαγαν ένα παίγνιο διαμόρφωσης άποψης στηριγμένο πάνω στο μοντέλο Friedkin Johnsen [17]. Στο παίγνιο αυτό κάθε πράκτορας i είναι ένας *εγωιστικός πράκτορας* του οποίου η στρατηγική του είναι η άποψη x_i που δημόσια εκφράζει. Για ένα συγκεκριμένο διάνυσμα δημοσίων απόψεων $x = (x_1, \ldots, x_n)$, ο πράκτορας i λαμβάνει ένα κόστος διαφωνίας $C_i(x_i, x_{-i})$, όπου

$$C_i(x_i, x_{-i}) = \sum_{j=1}^{n} w_{ij}(x_i - x_j)^2 + w_i(x_i - s_i)^2$$

Μέσα από αυτό το παιγνιοθεωρητικό πρίσμα, το μοντέλο Friedkin Johnsen είναι το *simultaneous best response dynamics* του παραπάνω παίγνιου. Πιο συγκεκριμένα ας υποθέσουμε οτι οι πράκτορες παίζουν το παραπάνω παιχνίδι σε γύρους. Αν ο κάθε πράκτορας διαλέγει ως άποψη την άποψη με το μικρότερο κόστος διαφωνίας βάσει των δημοσίων απόψεων των άλλων πρακτόρων στον προηγούμενο γύρο, τότε προκύπτει το μοντέλο Friedkin Johnsen. Επίσης το σταθερό του σημείο $x^* \in [0,1]^n$ είναι η ισορροπία Nashτου παραπάνω παίγνιου.

Αξίζει να επισημανθεί ότι οι Bindel, Kleinberg και Oren εισήγαγαν ένα γενικότερο πλαίσιο για την μελέτη των δυναμικών διαμόρφωσης άποψης. Το πλαίσιο αυτό υποδεικνύει ότι η διαδικασία σχηματισμού άποψεων μπορεί να περιγραφεί ως μία *δυναμική* ενός παιγνίου διαμόρφωσης άποψης. Αυτό το πλαίσιο είναι γενικότερο καθώς διαφορετικές πτυχές των διαδικασίων διαμόρφωσης απόψης μπορούν εύκολα να μοντελοποιηθούν με την ορισμό κατάλληλων παιγνίων. Επιπλέον το πλαίσιο αυτό επιτρέπει την μελέτη της δυναμικής συμπεριφοράς, γνωστών παιγνιοθεωρητικών στρατηγικών (best response, no-regret, fictitious play) σε τέτοιου είδους παίγνια.

Το μοντέλο Friedkin Johnsen έχει μελετηθεί εκτενώς τα τελευταία χρόνια. Όπως έχουμε ήδη αναφέρει στην εργασία [81] αποδείχθηκε η ύπαρξη ενός μοναδικού σταθερού σημείου x^* και ο *γραμμικός ρυθμός* σύγκλισης του μοντέλου. Στην εργασία [17], όπου εισήχθη το αντίστοιχο παίγνιο διαμόρφωσης άποψης, ποσοτικοποιήθηκε η αναποτελεσματικότητα της ισορροπίας Nash σε σχέση με το βέλτιστο συνολικό κόστος διαφωνίας. Αποδείχθηκε ότι το *Τίμημα της Αναρχίας* είναι $9/8$ στην περίπτωση όπου $w_{ij} = w_{ji}$. Αποδείχθηκαν επίσης άνω φράγματα για το *Τίμημα της Αναρχίας* την περίπτωση των αβαρών κατευθυνόμενων γραφημάτων Euler. Σε πιο πρόσφατες ερευνητικές εργασίες [15, 47, 38] επέκτάθηκαν τα άνω φραγματα σε άλλες οικογένειες γραφημάτων και σε πιο γενικές συναρτήσεις κόστους διαφωνίας. Στις εργασίες [140, 70, 16] μελετήθηκαν *διακριτές* παραλλαγές του μοντέλου Friedkin Johnsen στις οποίες οι πράκτορες μπορούν να υιοθετήσουν ως απόψης είτε το 0 ή 1 και εξετάστηκαν οι ιδιότητες σύγκλισης τους. Μία πρόσφατη γραμμή έρευνας μελετά συνδυαστικά προβλήματα σχετικά με την τροποποίηση του σταθερού σημείου x^* του μοντέλου Friekdin Johnsen [82, 1, 114].

Το μοντέλο Hegselmann Krause

Η πόλωση των απόψεων είναι ένα πολύ σύνηθες κοινωνικό φαινόμενο. Συχνά οι άνθρωποι σχηματίζουν *ομάδες απόψεων* όπου μέλη της ίδιας ομάδας μοιράζονται σχεδόν την ίδια άποψη, ενώ άτομα από διαφορετικές ομάδες έχουν αρκετά διαφορετικές απόψεις. Ο λόγος αυτής της πόλωσης είναι αρκετά απλός: *Άτομα με παρόμοιες απόψεις τείνουν να αναπτύσσουν κοινωνικές σχέσεις. Ενώ άτομα με εντελώς διαφορετικές απόψεις τείνουν διακόπτουν τις σχέσεις τους.*

Το 2002 οι Hegselmann και Krause πρότειναν ένα μοντέλο για τη διαμόρφωση απόψεων που ενσωμάτωσε αυτές τις ιδέες με έναν πολύ απλό τρόπο: σε κάθε γύρο ο κάθε πράκτορας υιοθετεί ως νέα άποψη το μέσο όρο της τρέχουσα άποψης του και των απόψεων των άλλων πρακτότων που είναι σε μικρή απόσταση από την δική του άποψη [89]. Το πόσο μικρή χρειάζεται να είναι αυτή η απόσταση ποσοτικοποιείται από την θετική σταθερά $\varepsilon > 0$.

Το μοντέλο Hegselmann Krause

1: n πράκτορες.
2: $x_i(0) \in [0, 1]$, η αρχική άποψη του πράκτορα i.
3: Στο γύρο $t \geq 1$ κάθε πράκτορας i ανανεώνει την άποψη του:

$$x_i(t) = \frac{\sum_{j \in N_i(t)} x_j(t-1) + x_i(t-1)}{|N_i(t)| + 1}$$

όπου $N_i(t) = \{j \neq i : \ |x_i(t-1) - x_j(t-1)| \leq \varepsilon\}$

Το μοντέλο Hegselmann Krause έχει άπειρα σταθερά σημεία: κάθε διαμέριση των πρακτόρων σε συμπλέγματα γνωμών (*opinion clusters*) με απόσταση μεγαλύτερη από ε, είναι ένα σταθερό σημείο. Παρόλο που η απόδειξη ύπαρξης σταθερών σημείων είναι πολύ απλή, δεν είναι καθόλου σαφές αν το σύστημα φτάνει ποτέ σε ένα τέτοιο σταθερό σημείο. Λόγω των αποτελεσμάτων των εργασιών [91, 105, 113] γνωρίζουμε πως το σύστημα σταθεροποιείται σε κάποια χρονική στιγμή. Σε μετέπειτα ερευνητικές δουλείες [107, 135, 14] παρέχονται άνω φράγματα για τον αριθμό των γύρων που απαιτούνται για τη σύγκλιση. Τα τελευταία αποτελέσματα δείχνουν το συστημα χρειάζεται το πολύ $O(n^3)$ γύρους για να σταθεροποιηθεί [14]. Πιο πρόσφατα, αποδείχθηκε ότι υπάρχουν περιπτώσεις στις οποίες το μοντέλο Hegselmann Krause χρειάζεται τουλάχιστον $\Omega(n^2)$ γύρους για να φτάσει σε σταθερό σημείο [139].

Το μοντέλο Hegselmann Krause έχει προσελκύσει το ενδιαφέρον διάφορων επιστημονικών κλάδων όπως η Θεωρητική Πληροφορική, η Στατιστική Φυσική και η Επιχειρισιακή Έρευνα. Οι επιστημονικές εργασίες που αφορούν σε παραλλαγές και γενίκευσεις του μοντέλου Hegselmann Krause είναι τόσο πολλές που αναφέρουμε μόνο τις άμεσα σχετιζόμενες με αυτή τη διατριβή. Στις εργασίες [105, 91, 35] παρέχονται αποτελέσματα σύγκλισης για γενικεύσεις του μοντέλου. Στην εργασία [37] αποδεικνύεται πως μια γενικευμένη εκδοχή του μοντέλου Hegselmann Krause με χρονικά αμετάβλητους πράκτορες συγκλίνει σε σταθερό σημείο. Επιπλέον έχει πραγματοποιηθεί σημαντική πειραματική έρευνα σχετικά με τις ιδιότητες σύγκλισης παραλλαγών του μοντέλου Hegselmann Krause, καθώς και με τις τιμές της παραμέτρου ε, που εγκυόνται την σύγκλιση των απόψεων σε ομοφωνία [72, 106].

Παρουσίαση Προβλημάτων της παρούσας Διδακτορικής Διατριβής

Στο υπόλοιπο του κεφάλαιου παρουσιάζονται τέσσερις επεκτάσεις των μοντέλων Friedkin Johnsen και Hegselmann Krause που μελετήθηκαν στην παρούσα διατριβή. Στις αντίστοιχες ενότητες παρουσιάζεται η σημασία και τα σχετικά αποτελέσματα της κάθε επέκτασης.

Τυχαιοκρατικά Παίγνια Διαμόρφωσης Άποψης

Όπως έχουμε ήδη αναφέρει τόσο το μοντέλο Friedkin Johnsen όσο και το αντίστοιχο παίγνιο, είχαν τεράστια επιρροή στην μελέτη των δυναμικών διαμόρφωσης άποψης. Ωστόσο υπάρχουν σημαντικές περιπτώσεις όπου το μοντέλο Friedkin Johnsen δεν περιγράφει κατάλληλα τη δυναμική των απόψεων, λόγω του μεγάλου ποσού πληροφοριών που απαιτεί να ανταλάσσουν οι πράκτορες. Πιο συγκεκριμένα, σε κάθε γύρο ο κανόνας ανανέωσης

$$x_i(t) = \frac{\sum_{j \neq i} w_{ij} x_j(t-1) + w_i s_i}{\sum_{j \neq i} w_{ij} + w_i}$$

απαιτει από τον κάθε πράκτορα να μαθαίνει *όλες* τις απόψεις των πρακτόρων που τον επηρεάζουν, δηλαδή όλων των πρακτόρων j με $w_{ij} > 0$. Στα σημερινά μεγάλα κοινωνικά δίκτυα όπου οι χρήστες έχουν συνήθως αρκετές εκατοντάδες φίλους, η υπόθεση ότι κάθε μέρα ο κάθε χρήστης μαθαίνει τις απόψεις όλων των κοινωνικών του επαφών, είναι μη ρεαλιστική. Σε τέτοιες περιπτώσεις μια πολύ πιο λογική υπόθεση είναι ότι τα άτομα συναντούν τυχαία ένα μικρό υποσύνολο των γνωστών τους και αυτές είναι οι μοναδικές απόψεις που μαθαίνουν.

Για να μοντελεποίησουμε τις παραπάνω ιδέες, θεωρούμε μια *τυχαιοκρατική παραλλαγή* του παίγνιου διαμόρφωσης άποψης που εισήχθη στο [17]. Στην παραλλαγή μας, για ένα δεδομένο διάνυσμα δημοσίων απόψεων $x = (x_i, x_{-i}) \in [0,1]^n$, το κόστος διαφωνίας του παίκτη i είναι η παρακάτω τυχαία μεταβλητή $C_i(x_i, x_{-i})$:

- Ο παίκτης i συναντάει τυχαία **μόνο ένα** γείτονα του j με πιθανότητα,

$$p_{ij} = \frac{w_{ij}}{\sum_{j \neq i} w_{ij}}$$

- Ο παίκτης i βιώνει κόστος διαφωνίας

$$C_i(x_i, x_{-i}) = (1 - \alpha_i)(x_i - x_j)^2 + \alpha_i (x_i - s_i)^2,$$

  όπου $\alpha_i = w_i / (\sum_{j \in N_i} w_{ij} + w_i)$.

Το πραπάνω τυχαιοκρατικό παίγνιο διαμόρφωσης άποψης, βασίζεται στην κοινή πεποίθηση ότι η επιρροή μεταξύ δύο άτομων σε μια κοινωνία είναι η συχνότητα αλληλεπίδρασης των ατόμων αυτών. Η ισορροπία Nash αυτού του παιγνίου (που ορίζεται σε σχέση με το αναμενόμενο κόστος διαφωνίας) είναι ακριβώς η ίδια με την ισορροπία Nash του παιγνίου διαμόρφωσης άποψης που εισήχθη στο [17]. Επιπλέον, το *simultaneous best response dynamics* (σε σχέση με το αναμενόμενο κόστος διαφωνίας) για το τυχαιοκρατικό παίγνιο διαμόρφωσης άποψης είναι πάλι το μοντέλο Friedkin Johnsen.

Η συνεισφορά μας

Μελετάμε τις ιδιότητες σύγκλισης *φυσικών και αποδοτικών δυναμικών* σε αυτό το τυχαίοκρατικό παίγνιο διαμόρφωσης άποψης. Με τον όρο *φυσικές* εννοούμε ότι οι πράκτορες ανανεώνουν τις δημόσιες απόψεις τους στην προσπάθειά τους να ελαχιστοποιήσουν το κόστος διαφωνίας που βιώνουν. Με τον όρο *αποδοτικές* εννοούμε ότι ο κανόνας ανανέωσης των απόψεων σέβεται τους περιορισμούς στην ανταλλαγή πληροφορίας που θέτει το παίγνιο μας. Δηλαδή σε κάθε γύρο ο κάθε πράκτορας μαθαίνει *μόνο* τη γνώμη του πράκτορα που συνάντησε τυχαία. Για παράδειγμα,

- Το μοντέλο Friedkin Johnsen είναι φυσικό: *κάθε πράκτορας επιλέγει τη γνώμη που ελαχιστοποιεί το αναμενόμενο κόστος διαφωνίας.*
- Το μοντέλο Friedkin Johnsen δεν είναι αποδοτικό: *για να υπολογίσει αυτή τη άποψη ο πράκτορας πρέπει να γνωρίζει τις απόψεις όλων των πρακτόρων με* $w_{ij} > 0$.

Στο Κεφάλαιο 3, παρουσιάζουμε ένα κανόνα ανανέωσης που οδηγεί σε μια φυσική και αποδοτική δυναμική των απόψεων. Αυτός ο κανόνας ανανέωσης απαιτεί ότι ο κάθε πράκτορας μαθαίνει μόνο την άποψη του πράκτορα που συνάντησε τυχαία και αυτό καθιστά την παραγόμενη δυναμική αποδοτική. Την ίδια στιγμή ο ίδιος κανόνας ανανέωσης εξασφαλίζει την ιδιότητα *no-regret* στον κάθε πράκτορα. Με άλλα λόγια εξασφαλίζει πως το κόστος διαφωνίας που ο κάθε πράκτορας βιώνει κατά την διάρκεια του παιγνίου είναι το ελάχιστο δυνατό. Αυτό καθιστά τον κανόνα μας μια φυσική επιλογή προς εγωιστές πράκτορες που ενδιαφέρονται μόνο για το ατομικό κόστος διαφωνίας που βιώνουν. Δείχνουμε ότι αν ο κανόνας υιοθετηθεί από όλους τους πράκτορες, τότε οι απόψεις των πρακτόρων είναι ε-κοντά στην ισορροπία Nash του παιγνίου σε $\widetilde{O}(1/\varepsilon^2)$ γύρους.

Στη συνέχεια, διερευνούμε την ύπαρξη άλλων κανόνων ανανέωσης που εξασφαλίζουν την *no-regret* ιδιότητα στο κόστος διαφωνίας των παικτών, ενώ ταυτόχρονα η παραγώμενη δυναμική των απόψεων συγκλίνει με γρηγορότερο ρυθμό στην ισορροπία Nash του παιγνίου. Στην προσπάθεια απάντησης αυτού του ερωτήματος ανακαλύψαμε ένα πολύ ενδιαφέρον φαινόμενο. Αν ο κανόνας ανανέωσης εξασφαλίζει την *no-regret* ιδιότητα, τότε η παραγόμενη δυναμική χρειάζεται τουλάχιστον $\Omega(1/\varepsilon)$ γύρους για να είναι ε-κοντά στην ισορροπία Nash, ενώ το παραπάνω αποτέλεσμα δεν ισχύει για κανόνες ανανέωσης που δεν εξασφαλίζουν την *no-regret* ιδιότητα. Χρησιμοποίωντας πρόσφατες stochastic gradient μεθόδους [94, 131, 13] καταφέραμε να σχεδιάσουμε ένα κανόνα ανανέωσης που δεν εξασφαλίζει την *no-regret* ιδιότητα αλλά η παραγόμενη δυναμική των απόψεων συγκλίνει σε $O(\log^2(1/\varepsilon))$ γύρους στην ισσοροπια Nash.

Παίγνια διαμόρφωσης άποψης με συνάθροιση και αρνητική επιρροή

Ένα άλλο σημείο κριτικής του μοντέλου Friedkin Johnsen είναι ότι αγνοεί επιρροές στους πράκτορες που προέρχονται από *καθολικές ιδιότητες* των δημοσίων απόψεων. Σε πολλούς τομείς, οι δημόσιες απόψεις των πολιτών δεν επηρεάζονται μόνο από αλληλεπιδράσεις με τον κοινωνικό τους κύκλο και τις προσωπικές τους πεποιθήσεις, αλλά και από το σύνολο των δημοσίων απόψεων στην κοινωνία. Για παράδειγμα οι άνθρωποι συχνά εκτίθενται σε παγκόσμιες τάσεις, σε κοινωνικά πρότυπα, σε αποτελέσματα εκλογών κ.λ.π. Επιπλέον, πολλές φορές ομάδες ατόμων πρέπει να συμφωνήσουν σε μια κοινή δράση, ακόμη και αν οι πεποιθήσεις τους και οι απόψεις είναι εντελώς διαφορετικές.

Για να μοντελοποίησουμε τέτοιες καταστάσεις θεωρούμε ένα κανόνα συνάθροισης (*aggregation rule*), ο οποίος αποτυπώνει τις απόψεις του κοινού σε μια *ενιαία κοινωνική άποψη* που αντιπροσωπεύει την γενική άποψη για ένα συγκεκριμένο ζήτημα. Οι πράκτορες αναμένουν τον αντίκτυπο της δημόσιας άποψης τους στην διαμόρφωση αυτής της *ενιαίας κοινωνικής άποψης* και το λαμβάνουν υπόψιν στην δημόσια άποψη που εκφράζουν. Έτσι εξετάζουμε μια γενίκευση των παίγνιων διαμόρφωσης άποψης που εισήχθησαν στο [17] στην οποία για ένα δεδομένο διάνυσμα δημοσίων απόψεων $x = (x_i, x_{-i})$ ο πράκτορας i βιώνει κόστος διαφωνίας

$$C_i(x) = \sum_{j \neq i} w_{ij}(x_i - x_j)^2 + w_i(x_i - s_i)^2 + \alpha_i(aggr(x) - s_i)^2 \ .$$

όπου $aggr : [0,1]^n \mapsto [0,1]$ είναι η συνάρτηση που εξάγει την *ενιαία κοινωνική άποψη* από το σύνολο των απόψεων του κοινού. Ο όρος $\alpha_i \geq 0$ ποσοτικοποιεί την επιροή που ασκεί η *ενιαία κοινωνική άποψη* στον πράκτορα i.

Με βάσει προηγούμενες εργασίες πάνω στο *wisdom of crowds* [92, 83], επικεντρωνόμαστε στην περίπτωση που ο κανόνας συνάθροισης $aggr(x)$ είναι ο μέσος όρος των δημοσίων απόψεων των πρακτόρων, δηλαδή

$$aggr(x) = \sum_{j=1}^{n} x_j/n$$

Η συνεισφορά μας

Στο κεφάλιαο 4 μελετάμε τα παίγνια συνάθροισης που παρουσιάσαμε παραπάνω. Αξίζει να σημειωθεί πως τα παίγνια διαμόρφωσης άποψης που εισήχθησαν στο [17], είναι ειδική περίπτωση των παίγνιων συνάθροισης όπου όλοι οι συντελεστές $\alpha_i = 0$. Με μια πρώτη ματιά, ο πρόσθετος όρος συνάθροισης φαίνεται να μην έχει σημαντικό αντίκτυπο στις ιδιότητες του παιγνίου. Αυτό απέχει πολύ από την αλήθεια! Με την παρουσία αυτού του απλού όρου, τόσο οι ιδιότητες σύγκλισης όσο και τα άνω φράγματα στο *Τίμημα της Αναρχίας* είναι υπό σοβαρή αμφισβήτηση. Σε γενικές γραμμές, αυτό συμβαίνει επειδή ο όρος αυτός εισάγει αρνητική επιρροή μεταξύ των πρακτόρων.

Λόγω της εισαγόμενης αρνητικής επιρροής, οι πράκτορες ενδέχεται να θελήσουν να υιοθετήσουν απόψης εκτός του διαστήματος $[0, 1]$. Αν και αυτό δεν αποτελεί εκ των προταίρων μια κακή υπόθεση, υπάρχουν περιπτώσεις όπως οι εκλογές στις οποίες οι απόψεις αναγκαστικά βρίσκονται σε ένα καθορισμένο εύρος. Για να καλύψουμε όλες τις περιπτώσεις, θεωρούμε τόσο την *unresctricted case* όπου οι πράκτορες μπορούν να επιλέξουν ως γνώμη οποιοδήποτε πραγματικό αριθμό και την *resctricted case* στην οποία οι πρακτόρες ανακαστικά πρέπει να επιλέξουν μία άποψη στο διάστημα $[0, 1]$. Και στις δύο περιπτώσεις παρουσιάζουμε αποτελέσματα τόσο για τις ιδιότητες σύγκλισης του *simultaneous best response dynamics* όσο και άνω φράγματα για το *Τίμημα της Αναρχίας*.

Αποδεικνύουμε ότι κάτω από πολύ γενικές υποθέσεις σχετικά με τις τιμές των α_i, το *simultaneous best response* είναι ε-κοντά στην ισορροπία Nash σε $O(n^2 \log n/\varepsilon)$ γύρους. Το παραπάνω ισχύει τόσο για την *unresctricted case* όσο και την *resctricted case* . Δεδομένου ότι το *simultaneous best response* απαιτεί από τους πράκτορες να γνωρίζουν την *μέση δημόσια άποψη* σε κάθε γύρο, μια πληροφορία που είναι δύσκολο να αποκτηθεί, εξετάζουμε μια *outdated* έκδοχη του. Τώρα οι πράκτορες μαθαίνουν τις απόψεις των γειτόνων τους σε κάθε γύρο, αλλά η *μέση δημόσια άποψη* ανακοινώνεται σε αραιά χρονικά διαστήματα. Δείχνουμε ότι τα ίδια αποτελέσματα σύγκλισης παραμένουν ακόμη και σε αυτή την περίπτωση.

Στη συνέχεια, εστιάζουμε την προσοχή μας στην ποιότητα της ισορροπίας Nash σε σχέση με το βέλτιστο συνολικό κόστος διαφωνίας. Χρησιμοποιώντας την τεχνική Local Smoothness [128], δείχνουμε ότι στην *unresctricted case* το *Τίμημα της Αναρχίας* είναι $9/8 + O(\alpha/(wn^2))$ αν $w_i = w$ και $\alpha_i = \alpha$. Παρόλο που η *resctricted case* είναι πολύ πιο δύσκολο να αναλυθεί, δείχνουμε ότι το *Τίμημα της Αναρχίας* είναι το πολύ $3 + \sqrt{2}$ στην περίπτωση που $w_i = \alpha_i = 1$.

Network Hegselmann Krause model

Όπως έχουμε ήδη αναφέρει, το μοντέλο Hegselmann Krause είχε τεράστια επιρροή στη μελέτη των δυναμικών διαμόρφωσης άποψης. Ωστόσο το μοντέλο Hegselmann Krause υποθέτει έμμεσως κάτι μάλλον αμφισβητήσιμο. Σύμφωνα με το μοντέλο, δύο πράκτορες i, j ασκούν επιρροή ο ένας στον άλλον, κάθε φορά που έχουν παρόμοιες απόψεις, $|x_i(t) - x_j(t)| \leq \varepsilon$. Στην πράξη η ύπαρξη παρόμοιων απόψεων δεν επαρκεί για την αλληλεπίδραση δύο ατόμων καθώς και η ύπαρξη κάποιας κοινωνικής σχέσης είναι απαραίτητη.

Εισάγουμε μια πολύ απλή γενίκευση του μοντέλου Hegselmann Krause η οποία ενσωματώνει τα παραπάνω ζητήματα. Υποθέτουμε την ύπαρξη ενός μη κατευθυνόμενου γραφήματος $G = (V, E)$, όπου οι κόμβοι V αντιπροσωπεύουν τους πράκτορες και οι ακμές E τις κοινωνικές σχέσεις μεταξύ τους. Στη γενίκευση μας, που ονομάζεται Nework HK model, κάθε πράκτορας υιοθετεί ως νέα άποψη, το μέσο όρο της τρέχουσα άποψης του με τις απόψεις των γειτόνων του στο G που είναι ε-κοντά στην δική του.

Network Hegselmann Krause model

1: Μη κατευθυνόμενος γράφος $G = (V, E)$.
2: n πράκτορες.
3: $x_i(0) \in [0, 1]$, η αρχική άποψη του πράκτορα i.
4: Στο γύρο $t \geq 1$, κάθε πράκτορας i υιοθετεί ως άποψη:

$$x_i(t+1) = \frac{\sum_{j \in N_i(t)} x_j(t) + x_i(t)}{|N_i(t)| + 1}$$

όπου $N_i(t) = \{j \neq i : \ |x_i(t) - x_j(t)| \leq \epsilon \text{ και } (i, j) \in E\}$

Είναι εύκολο για κάποιον να δει πως το μοντέλο Hegselmann Krause είναι η ειδική περίπτωση του Network Hegselmann Krause model, όπου το γράφημα G είναι κλίκα. Όπως και το αρχικό μοντέλο Hegselmann Krause, ετσι και το Network Hegselmann Krause model έχει άπειρο αριθμό σταθερών σημείων ανεξαρτήτως της τοπολογίας του G. Ωστόσο, τα αποτελέσματα σύγκλισης του μοντέλου Hegselmann Krause δεν είναι εύκολο να γενικευθούν. Η απόδειξη σύγκλισης του μοντέλου Hegselmann Krause εξαρτάται σε μεγάλο βαθμό από την παρακάτω ιδιότητα: *η διάταξη των πρακτόρων από αριστερά προς τα δεξιά σύμφωνα με τις απόψεις τους παραμένει πάντα η ίδια*. Δυστηχώς αυτή η δομή ισχύει μόνο όταν το G είναι κλίκα και για αυτό η απόδειξη σύγκλισης του Network Hegselmann Krause χρειάζεται μια εντελώς διαφορετική προσέγγιση από τις μέχρι τώρα αποδείξεις σύγκλισης του κλασσικού μοντέλου Hegselmann Krause.

Στο κεφάλαιο 4 αποδεικνύουμε πως το Network Hegselmann Krause model συγκλίνει πάντα σε ένα σταθερό σημείο. Για την απόδειξη αυτή θεωρούμε την ακολουθία μη κατευθυνόμενων γραφημάτων που κωδικοποίουν τις αλλεπιδράσεις μεταξύ των πρακτόρων σε κάθε γύρο. Δηλαδή, τις ακμές του G στις οποίες οι δύο πράκτορες της ακμής, εκφράζουν απόψεις που είναι ε-κοντά. Στη συνέχεια χρησιμοποιούμε την έννοια του *weak connectivity*, που εισήχθη στο [96], για να αποδείξουμε είτε το συνολικό δυναμικό συστήμα χωρίζεται σε ανεξάρτητα υποσυστήματατα είτε ότι όλοι οι πράκτορες υιοθετούν την ίδια άποψη. Μερικά από τα αποτελέσματά μας συμπίπτουν με τα αποτελέσματα [91, 105] σχετικά με γινόμενα στοχαστικών πινάκων, ωστόσο η προσέγγιση μας είναι απλούστερη και περιλαμβάνει πιο απλές αποδείξεις.

Τυχαιοκρατικό μοντέλο Hegselmann Krause

Όπως συζητήσαμε προηγουμένως το μοντέλο Friedkin Johnsen είναι ακατάλληλο για τη μοντελοποίηση διαδικασιών διαμόρφωσης άποψης σε μεγάλα κοινωνικά δίκτυά, λόγω της μεγάλης ανταλλαγής πληροφορίας που υποθέτει. Από αυτή τη σκοπιά τα πράγματα είναι πολύ χειρότερα στο μοντέλο Hegselmann Krause. Τώρα κάθε πράκτορας πρέπει να μάθει τις απόψεις *όλων* των άλλων πρακτόρων προκειμένου να προσδιορίσει ποιοί από αυτούς έχουν άποψη ε-κοντά στην δική του. Όπως και στα *Τυχαιοκρατικά Παίγνια Διαμόρφωσης Άποψης* υποθέτουμε πως σε κάθε γύρο κάθε πράκτορας συναντά τυχαία k άλλους πράκτορες, τις απόψεις των οποίων μαθαίνει. Στην συνέχεια ανανεώνει, την άποψη του στο μέσο όρο των απόψεων των πρακτόρων που τυχαία συνάντησε και είναι ε-κοντά στην δική του άποψη.

Τυχαιοκρατικό μοντέλο Hegselmann Krause

1: n πράκτορες.
2: $x_i(0) \in [0,1]$, η αρχική άποψη του πράκτορα i.
3: Στο γύρο $t \geq 1$, κάθε πράκτορας i:
  4: διαλέγει k άλλους πράκτορες ομοιόμορφα τυχαία, $R_i(t) \subseteq [n]$.
  5: ανανεώνει την άποψη του $x_i(t)$,

$$x_i(t) = \frac{\sum_{j \in N_i(t)} x_j(t-1) + x_i(t-1)}{|N_i(t)| + 1}$$

όπου $N_i(t) = \{j \neq i : \ |x_i(t-1) - x_j(t-1)| \leq \varepsilon \text{ και } j \in R_i(t)\}$

Στο κεφάλαιο ;;, δείχνουμε ότι το τυχαιοκρατικό μοντέλο Hegselmann Krause φτάνει πάντα σε σταθερό σημείο. Όπως και στο Network Hegselmann Krause model, η ταξινόμιση των πρακτόρων από τα αριστερά προς τα δεξιά σύμφωνα με τις απόψεις τους, δεν διατηρείται στο χρόνο και ως εκ τούτου οι τεχνικές για την απόδειξη σύγκλισης του μοντέλου Hegselmann Krause δεν μπορούν να εφαρμοστούν. Το τυχαίοκρατικό μοντέλο Hegselmann Krause ενέχει την επίπρόσθετη δυσκολία της ασύμμετρης επιρροής. Στο τυχαίοκρατικό μοντέλο Hegselmann Krause ο πράκτορας i μπορεί να επηρεάζει τον πράκτορα j, ενώ ο j να μην επηρεάζει το πράκτορα i. Αξίζει να σημειωθεί πως τέτοια ασυμμετρία στην επιρροή μεταξύ των πρακτόρων δεν μπορεί να υπάρξει στο Network Hegselmann Krause model. Αν και τα δυναμικά συστήματα μέσου όρου που επιτρέπουν τέτοια ασυμμετρία στους συντελεστές είναι πολύ δύκολο να αναλυθουν [37, 15], στην περίπτωση του τυχαιοκρατικού Hegselmann Krause καταφέραμε να αποδειξουμε πως το σύστημα συγκλίνει σε σταθερό σημείο με μεγάλη πιθανότητα.

Chapter 2

Introduction

2.1 The Big Picture

This thesis lies on the intersection of algorithmic game theory, dynamical systems and convex optimization. These areas admit beautiful connections that have lead to many fertile results over the years. The «motivating umbrella» of our study comes from the world of *opinion formation*. The latter means that all the considered settings thay may be non-cooperative games, dynamical systems or even combinatorial optimization problems relate to proposed models on the way people form their opinions. Before deeping into the details of the opinion formation context, we briefly introduce the above mentioned connections and how they relate to our work.

Games, Equilibrium and Efficiency

The tremendous success of game theory is based on the fact that most aspects of everyday's life can be efficiently captured by appropriate *non-cooperative games*. In a non-cooperative game, each agent selects an action from a set of posible actions so as to maximize her payoff which is a function of her selected action and the selected actions of the others. The exact definition of the action sets and the payoff functions depends on each specific setting. For example in games modelling traffic networks, the action set of an agent is the set of paths from a destination node to a target node, while her cost[1] is the travel time that normally depends on the number of agents usings edges of her selected path. In games modelling the opinion formation process, each agent selects an opinion[2] so as to minimize a disagreement cost function that

[1]The payoff is the cost multiplied by -1.

[2]Typically an opinion is a number in $[0, 1]$.

also depends on the other agents' opinions. When all players have chosen actions such that they simultaneously maximize their payoff (none of them can increase her payoff by selecting a different action), then we say that the system has reached a *Nash Equilibrium*.

One can easily find games (even with 2 agents) in which Nash Equilibrium does not exist if the agents have to deterministically select their actions (pure strategies). However as John F. Nash proved in his celebrated theorem [116], at least one Nash Equilibrium exists if the agents are allowed to select their action according to a probability distribution over a finite action set (mixed strategies). Unfortunately computing such equilibria is a computationally hard task, since it was proven to be PPAD-complete even for the 2-agent case [55, 39].

In an attempt to understand how efficient Nash Equilibria are in terms of social payoff [3], Koutsoupias and Papadimitriou introduced the notion of *Price of Anarchy* [101]. Price of Anarchy is the ratio between the maximum payoff that the agents can acheive in total over the minimum total payoff acheived at a Nash Equilibrium. Unfortunately this ratio can be arbitrarily high in general, meaning that agents' selfish nature can result in very bad outcomes for the overall system. The need for designing systems that remain efficient even under the impact of selfishness, lead in huge line of reseach studying the inefficiency of equilibrium in various kind of games (see for example the very first representatives [129, 48] of this research line concerning the price of anarchy in congestion games). We follow this line through studying the price of anarchy in opinion formation games with respect to the social disagreement cost.

Game-playing Strategies and Natural Dynamics

Nash Equilibrium has a *static* nature in the sense that it describes a steady state of a multiagent system in which none is willing to deviate from. However in most interesting settings agents do not play the game once and for all, but they repeatedly play the same game over and over again e.g. the drivers of a town play the same congestion game every morning of the year. In such *dynamic settings* Nash Equilibrium does not provide answers neither to what a selfish agent should do in order to maximize her long-term payoff nor to what the dynamic behavior of the overall system will be.

The question on how agents should update their actions in order to maximize their long-term payoff is tremendously hard and does not admit

[3]Social payoff typically measures the total happiness of the agents at an equilibrium which is the total sum of the agents' payoffs. However in several contexts they are also other meaningful functions capturing the social payoff.

a concrete answer. However there are some widely accepted game-playing strategies based on the following natural principle: *agents select the action maximizing their payoff with respect to the past actions of the other agents.* The most intuitive forms of this principle is *fictitious play* proposed by Brown in 1951 [27] and the *best response strategy* [62]. In the first case, agents select the action that maximizes their payoff with respect to the actions of the others in the whole history of the play, whearas in the *best response strategy* the payoff - maximizing strategy is computed with respect to the actions of the others agents in the previous round.

Having determined what is *natural* for a selfish agent to do in repeatedly played games, the following question arises: If all agents update their according to fictitious play or best response, does the system converges to Nash Equilibrium? This very reasonable question has initialized a long line of research about the convergence properties of such dynamics in games. Probably the most celebrated result in this line of research dates back to the result of Robinson proving that *fictitious play* converges to mixed Nash Equilibrium in zero-sum games [125]. The study on the convergence properties of *best response dynamics* has mainly focused on an important class of games, called *congestion games*. This kind of games admit a potential function meaning than whenever an agents changes her action for an action with better payoff, the potential function increases by the same amount of the payoff-increase [127]. The latter implies that any local minimum of the potential function is also a Nash Equilibrium and that *best response dynamics* always converges to equilibrium. Although computing Nash Equilibrium in congestion games proved to be a computationally hard problem (PLS-complete) [68] meaning that *best response dynamics* can take exponentially many rounds before reaching an equilibrium, there are many positive results for its convergence properties to approximate Nash equilibrium [112, 49, 46, 31, 30]. Moreover *best response dynamics* is known to converge to Nash Equilibrium in polynomial number of rounds for many important special cases of congestion games [66, 123] or when the instance of the game is contaminated with random noise [65, 4, 22]. Following this line of research, we provide convergence results for both fictitious play and best response dynamics in various opinion formation games.

No-regret Dynamics and Online Convex Optimization

Although both fictitious play and best response dynamics are very natural behavioral assumptions for selfish agents, one can argue that they do not properly capture the behavior of fully rational agents. This critique is quite fair since neither fictitious play nor the best response strategy provide guarantees about the long-term payoff of the agents. This means that although a myopic agent may select her actions according to them, a perspicacious agent has no real reason to follow them. To this end a very important connection between convex optimization and dynamics in games is revealed.

Surprisigly enough for a wide class of games, an agent can select her actions according to algorithms develloped in the area of online convex optimization and that do provide guarantees on her experienced payoff. These guarantees hold no matter the way the other agents select their actions, while the requirements for such an algorithm to exist are quite mild; convex action set and convex payoff function [87]. The guarantees that such algorithms provide, do not relate to the optimal payoff that an agent could aquire by knowing the actions of the others up front and by selecting her best-responding action at each round of the game. Obviously this is far too good to be true! However such algorithms provide quite strong guarantees related to the payoff of the *best fixed action*, which are formally expressed with the notion of *regret*. The *regret* of an online convex optimization algorithm is the time-averaged difference between the algorithm's acquired payoof and the payoff of the *best fixed action*[4]. Algorithms with regret tending to zero as the rounds of the game increase, are called *no-regret*. We remark that although there are several *no-regret algorithms* (see [87] for an introduction to the online convex optimization framework), the existence of such algorithms is far from being trivial. In fact the first no-regret algorithm, the seminal *Hedge algorithm* proposed by Hannan in 1957 [85], was a huge scientific surprise that triggered a vast amount of interest towards the design of no-regret algorithm. The interested reader can find a tiny subset of such algorithms in [104, 142, 88, 21, 86, 69].

Apart from designing no-regret algorithms, the algorithmic game theory community develloped a vast interest towards understanding the *dynamic* behavior of systems in which agents play according to no-regret algorithms [67, 78, 20, 33, 108, 5, 6, 119, 122, 115, 134, 110, 73, 133, 54, 56, 52]. For example it is known that in n-person finite games, *no-regret dynamics* converge to Coarse Correlated Equilibrium [73, 78, 133]. In [67] it was shown that in a large class of games with infinite strategy spaces and concave utility functions (socially concave games), no-regret dynamics converge to Pure Nash Equilibrium. No-regret dynamics are also known to converge to the mixed Nash Equilibrium of zero-sum games [54, 56] and to *locally* converge to the mixed Nash Equilibrium of n-person finite *generic* games [52].

[4]The payoff of a fixed action is the aggregated payoff of the agent if she always played this specific action at all rounds of the game. Notice that the payoff of an action may differ from round to round since the other agents may change their action from round to round.

Following this line of research we study the convergence properties of *no-regret dynamics* in opinion formation games. We prove convergence to Nash Equilibrium when agents update their opinions according to a seminal class of no-regret algorithms, called *Follow the Regularized Leader*. We also provide lower bounds on the convergence rate of *no-regret dynamics*.

Dynamical Systems and Distributed Convex Optimization

There exists a mutual relation between discrete-time dynamical systems and distributed algorithms. The algorithmic design of distributed protocols can be based on simple dynamical systems [97, 95, 12], while at the same time algorithmic ideas and techniques can be applied in analyzing the behavior of dynamical systems [35, 14, 36]. Frequently these «algorithmic proofs of convergence» admit a convex optimization flavor. For example in [44, 43, 40, 45] dynamics in Fisher markets are analyzed through an equivalence with coordinate descent methods. Since the main focus of this thesis is about the study discrete-time dynamical systems that model the opinion formation process, the connection between dynamics and convex optimization is apparent in a great part of this work. The reason is that many considered opinion formation games admit a convex potential function and thus establishing convergence properties can be done via proving that the agents collectively find a minimum of the potential function. As a result, gradient-based methods develloped in the field of distributed convex optimization [13, 41] and stochastic gradient descent methods [94, 131] served as irreplaceable conceptual tools in our work.

Multi-stage Combinatorial Optimization and Convex Optimization

The use of convex optimization techiques in the design of efficient combinatorial optimization algorithms has been a tremendous success. Over 30 years *linear programming* and *semi-definite programming* are used in the design of approximation algorithms [137], while more recently gradient-descent methods were introduced in the design of competive online algorithms [28, 10]. These techniques seem to be the only way approach in a recent line of research studying «classical» combinatorial optimization problems with data that evolve over time [63, 3, 18]. In this kind of problems data admit different values from stage to stage and the goal is to produce a time-varying solution that is «relative stable»[5], while remaining efficient at each separate round. For example the authors in [63] study a *dynamic version* of the classical facility

[5] Typically this is measured with an additional switching cost quantifying the change of the solution from stage to stage

location problem in which the clients change positions from round to round. This type of problems admit a very harsh combinatorial structure and thus the only way for tackling them is via solving an appropriate convex program (typically a linear program) and then rounding the fractional solution.

We adopt this approach to solve a multistage combinatorial optimization problem related to the way a political party should select her public positions over time so as to efficiently cover a set of opinion-changing voters. The considered problem is a *dynamic version* of the classical k-median problem where the requests are located in the real line, but their positions change over time. We were able to provide a polynonial time algorithm that produces an optimal solution via solving an appropriate linear program and efficiently rounding the fractional solution on the time domain.

2.2 How Opinions are Formed?

The study on the way people form their opinions has a long history (see [92]). Opinion formation is a *dynamic process* in which socially connected people (family, friends, colleagues) exchange information and this leads to changes in their expressed opinions over time. Today, the advent of the internet and social media makes the study of opinion formation in large social networks even more important; realistic models of how people form their opinions by interacting with each other are of great practical interest for prediction, advertisement etc. In an attempt to formalize the process of opinion formation, several models have been proposed over the years [60, 79, 89, 59].

The common assumption underlying most of these models, which dates back to DeGroot [60], is that opinions evolve through a form of repeated averaging of information collected from the agents' social neighborhoods. Initially each agent holds a value that represents her initial opinion. At each round, all agents simulataneously average their opinion with the opinions of the other agents, leading to a *dynamics* of the opinions. The coefficients of this averaging rule may differ from agents to agent and in fact may change over time. We remark that the precise definition on how these coefficients are formed is defined in each specific model. This general modelling framework is summarized up next.

Averaging Framework

1: n agents.
2: $x_i(0) \in [0, 1]$, agent's i initial opinion.
3: At round $t \geq 1$, each agent i updates her opinion:

$$x_i(t) = \sum_{j=1}^{n} p_{ij}(t) x_j(t-1)$$

where $p_{ij}(t) \geq 0$ and $\sum_{j=1}^{n} p_{ij}(t) = 1$

Example 2.1. *In the DeGroot model the coefficients are time invariant, $p_{ij}(t) = p_{ij}$ [60]. In the Hegselmann Krause model each agent averages her opinion with the opinions that are at distance at most 1 from her current opinion [89].*

At a first glance such averaging systems may seem naive and thus incapable of modelling complex natural processes such as the opinion formation. This is far from being true! There exists both empirical and theoretical evidence indicating that averaging systems admit a lot of expressive power.

From the empirical point of view, a strong indication about this expressive power is that diverse phenomena are efficiently captured by averaging systems similar in spirit with the ones describing the opinion formation process. Notable examples come from collective animal behavior such as bird flocking [36, 124], fish schooling [118, 121] and firefly flashings [111]. Other interesting applications include the aggregation of measurements in sensor networks data, the evolution of cell populations and the coordination of heart pacemaker cell signals [34]. Moreover experimental studies on the formed opinions of villagers in India about the price of the crops, have verified the predictive power of these opinion formation models [7].

From the theoretical point of view, this expressive power is indicated by the fact that *such systems can solve computational problems!* For example in [12], the community detection problem is solved through the use of a distributed algorithm based on a simple averaging system. Moreover such averaging systems can even simulate Turing machines [34]! Other interesting algorithmic applications of averaging systems can be found in [95, 97].

This wide range of applications has created an intense scientific interest towards the convergence properties of such averaging systems [35, 14, 117, 93, 29, 81, 23, 25, 103, 105]. More precisely, this line of research tries to shed light on the following question:

When do such averaging systems converge to stable points?

Unfortunately the above question does not admit a concrete answer. Such averaging models are analyzed more or less in an ad-hoc way and the ideas and techniques may substantially differ. Although there are results characterizing the convergence properties for classes of averaging systems [91, 105, 35], there is not a unified theory describing their dynamic behavior. In fact slight variations on the models may lead to totally different convergence properties.

This thesis mainly focuses on the convergence properties of generalizations of the Friedkin Johnsen model and the Hegselmann Krause model which are averaging systems modelling the opinion formation process. Both the FJ model and the HK model were seminal in the *opinion dynamics* literature and their convergence properties have been extensively studied (both of them are known to converge to stable points relatively fast). We study several natural extensions of the above-mentioned models incorporating issues and limitations arising on the way people form opinions and that have been disregarded by the original models. Our extensions render the previous known results inapplicable and thus our work contributes in further understanding their properties.

2.3 Friedkin Johnsen Model and Opinion Formation Games

One of the most influential models for opinion formation is the one proposed by Friedkin and Johnsen in 1990 [79]. The FJ model was initially proposed as a variant of the DeGroot model, capturing the fact that consensus on the formed opinions of a social group is rarely reached.

According to FJ model each person i has a public opinion $x_i \in [0, 1]$ and an internal opinion $s_i \in [0, 1]$, which is private and invariant over time. There also exists a weighted graph $G(V, E, w)$ representing a social network. The set of nodes V stands for the agents and the set of edges E for their social relations. The weight w_{ij} of an edge $(i, j) \in E$ is assumed to be positive $w_{ij} \geq 0$ and quantifies the influence that agent j poses on agent i. Finally each agent i admits a positive weight $w_i > 0$ that measures the confidence of the agent to her internal opinion. Initially, all nodes start with some public opinions and at each round t, update their public opinion $x_i(t)$ to the weighted average of the public opinions of their neighbors and their internal opinion.

Friedkin Johnsen model

1: A weighted graph $G(V, E, w)$.
2: $s_i \in [0, 1]$, agent's i internal opinion.
3: $x_i(0) \in [0, 1]$, agent's i initial opinion.
4: $w_i > 0$, agent's i confidence to her internal opinion.
5: At round $t \geq 1$ each agent i updates her opinion:

$$x_i(t) = \frac{\sum_{j \neq i} w_{ij} x_j(t-1) + w_i s_i}{\sum_{j \neq i} w_{ij} + w_i}$$

The FJ model has a very simple update rule, making it plausible for modeling natural behavior and its basic assumptions are aligned with empirical findings on the way opinions are formed [2, 102, 7]. At the same time, it admits a unique stable point x^* to which it converges with a *linear* rate no matter the initial opinions [81].

In their seminal work Bindel, Kleinberg and Oren introduced a game theoretic viewpoint of the FJ model [17]. They interpreted its update rule as the minimizer of a quadratic disagreement cost function and based on it they defined the following opinion formation game: Each agent i is a *selfish agent* whose strategy is the public opinion x_i that she expresses, incurring her a disagreement cost

$$C_i(x_i, x_{-i}) = \sum_{j \neq i} w_{ij}(x_i - x_j)^2 + w_i(x_i - s_i)^2 \tag{2.1}$$

Under this perspective, FJ model is the *simultaneous best response dynamics* and its stable point x^* is the unique Nash Equilibrium of this game. We remark that in [17], a more comprehensive framework for modelling the opinion formation process was introduced. Instead of modelling the opinion formation as a precise dynamical process, one can capture the exact same aspects by an appropriate opinion formation game. The evolvement of the opinions over time can be modelled as the dynamic behavior of the selfish agents when iteratively play such an opinion formation game. This modelling approach offers a fruitful level of abstraction since various opinion dynamics (for the same opinion formation game) can be produced by considering *natural game-playing strategies* such as *best response dynamics, no regret dynamics, fictitious play* etc.

There exists a large amount of literature concerning the FJ model. In [81] it was proven that FJ model always admits a unique stable point to which it converges with *linear rate* no matter the initial public opinions. In [17] where the respective opinion formation game was introduced, they quantified the inefficiency of Nash Equilibrium with respect to the total disagreement cost. They proved that the *Price of Anarchy* is $9/8$ in case $w_{ij} = w_{ji}$. They also provided PoA bounds in the case of unweighted Eulerian directed graphs. Latter works [15, 47, 38] extended the PoA bounds to other graph families and to more general disagreement cost functions. [140, 70, 16] introduced variants of the FJ model in which the strategy space of the agents is either 0 or 1 (capturing binary opinion settings such as referendums) and examine their convergence properties. In [32] a variant of the FJ model is examined, where each agent selects her public opinion so as to minimize the maximum distance of her internal opinion and the opinions of her neighbors. Moreover the social neighbors are not *static*, but depend on the expressed public opinions. Another recent line of research concerns combinatorial problems for influencing the stable point of the FJ model [82, 1, 114].

2.4 Hegselmann Krause Model

A very common social phenomena is the so-called *opinion polarization*. Frequently people form *opinion groups* in which members of the same group share almost the same opinion, whereas opinions of members of different groups are quite far away. The reason for this polarization is fairly simple: *People with similar opinions tend to develop social relations. At the same time, people with totally different opinions interrupt their relations.*

In 2002 Hegselmann and Krause proposed a model for opinion formation that captures this general intuition in a very straightforward way: at each round each agent averages her opinion with the opinions close to hers [89]. More specifically, the HK model assumes the existence of n agents each one of which as an initial opinion $x_i(0) \in [0, 1]$. At each round, each agent averages her current opinion with the opinions of the other agents within distance $\varepsilon > 0$. The parameter ε denotes how eager the agents are towards adopting different opinions.

The Hegselmann Krause model

1: n agents.
2: $x_i(0) \in [0, 1]$, agent's i initial opinion.
3: At round $t \geq 1$ each agent i updates her opinion:

$$x_i(t) = \frac{\sum_{j \in N_i(t)} x_j(t-1) + x_i(t-1)}{|N_i(t)| + 1}$$

where $N_i(t) = \{j \neq i : \ |x_i(t-1) - x_j(t-1)| \leq \varepsilon\}$

The HK model admits an infinite number of stable points: any partition of the agents to *opinion clusters* with distance greater than ε is a *stable point*. Moreover the HK model always reaches such an opinion cluster in finite time [91, 105, 113] and there are also upper bounds on the number of rounds needed for this to happen [107, 135, 14]. The state of the art result due to Bhattacharyya et al. is $O(n^3)$ [14]. More recently, it was shown that there are instances in which HK model needs at least $\Omega(n^2)$ rounds in order to converge [139], while closing this gap remains an interesting open question.

The HK model has attracted the attention of different scientific comminities such as theoretical computer science, physics, operation research and control theory. The amount of scientific work concerning variants and generalizations of the HK model is so large that we list the results most relavant to this thesis. The authors in [105, 91, 35] provide convergence results for generalizations of the HK model. In [37] it was proven that a generalized version of the HK model with partially stubborn agents, converges to equilibrium. Moreover, there has been significant experimental work on the convergence properties on variants of the HK model and on confidence levels that are sufficient or necessary for consensus [72, 106].

2.5 Problems Considered in this Thesis

In this section we present the problems considered in this thesis. The major part of our work concerns generalizations of the previously presented Friedkin Johnsen and Hegselmann Krause model. In Sections 2.5.1 and 2.5.2 we introduce our generalizations of the Friedkin Johnsen model, while in Sections 2.5.3 and 2.5.4 we introduce our generalizations of the Hegselmann Krause model. In Section 2.5.5, we present our results concerning the *facility reallocation problem* [58], which is a *dynamic version* of the well-studied *k-median problem* in which the requests change positions over time. Although this problem may seem quite out of context with respect to the topic of opinion dynamics and opinion formation games, it admits a very natural motivation on how political parties should assign public opinions to their members so as to efficiently represent voters with dynamically changing opinions over time. Before presenting each considered setting, we briefly discuss the common framework of *imperfect information* that is present in all the considered opinion dynamics and formation games and the connection of our results with ideas and techniques of convex optimization.

Imperfect Information

A recent line of research studies multiagent systems in settings where agents act under *imperfect information*. The latter means that the agents may not be aware of the overall state of the system and may have to decide their actions according to a small piece of information revealed to them. A very illustrative example, clarifying the notion of *imperfect information*, comes from the world of traffic networks. In the *classical* game-theoretic way of modelling, agents are assumed to play a congestion game where they select a path in the network based on the congestion of the paths in the previous round [98]. Here the following information exchange assumption is made: *the agents learn at the end of each round the congestion of all paths*. But how reasonable is this? In most practical settings, an agent only learns the congestion of the path that she selected and uses only this information to select a new path in the next round [99]. Obviously the exact form of *imperfect information* depends on the information exchage constraints that each specific setting poses and may take different forms from setting to setting [90, 53, 109, 26, 52, 98, 99].

In the context of opinion dynamics and opinion formation games, *imperfect information* takes a very natural and concrete form. When an opinion formation model (such as the FJ model or the HK model) assumes that an agent averages her opinion with the opinions of some other agents, it implicitly assumes that a social interaction among them was performed (the agents met, dicussed etc.). The problem is that such interactions usually come with a cost in realistic settings and this has been ignored by the proposed opinion models to greater or lesser extend. As a result, *imperfect information* in the context of opinion formation process means that an agent learns a limited amount of opinions of other agents, possibly much fewer than the opinions of her overall social circle. We remark that our extensions are motivated by various natural social phenomena, however all the examined opinion dynamics respect the above information exchange constraints.

Opinion Dynamics through Convex Optimization

The ideas and techniques develloped in the context of convex optimization proved to be a very powerful tool for many of the results that we subsequently present. Although the most straightforward application of these techniques appears in the design of a polynomial time algorithm for the *facility reallocation problem* (see Section 2.5.5), the most interesting ones come up in Sections 2.5.1 and 2.5.2. In these sections the convergence properties of extensions of the FJ model are analyzed through the use of through the use of recent gradient descent methods.

The relation between the FJ model and gradient descent methods comes out in various levels. At first, a step of the FJ model is equivalent to a step of the Newton method [24] applied to the quadratic function

$$\Phi(x_i, x_{-i}) = \sum_{(i,j)\in E} w_{ij}(x_i - x_j)^2 + \sum_{i\in V} w_i(x_i - s_i)^2 \tag{2.2}$$

which was identified by Bindel, Kleinberg and Oren as a *potential function* of their game [17]. As a result, one can prove that FJ model converges to Nash Equilibrium by proving that Netwon method with unit step size converges to the unique minimizer of the convex potential function $\Phi(x_i, x_{-i})$. The equivalence between converging to equilibrium and minimizing a potential function via a gradient descent method appears in all of our convergence results presented in Sections 2.5.1 and 2.5.2. For example in Section 2.5.2, we were able to identify sufficient conditions for convergence in a generalization of the FJ model with negative influences among the agents by requiring the convexity of an appropriate potential function. Moreover in Section 2.5.1, we used techniques develloped in the context of stochastic gradient descent [94, 131] and distributed gradient descent [13, 41] to analyze variants of the FJ model in imperfect information settings, where agents learn a small random subset of the opinions or have some outdated knowledge about the opinions of their friends. Finally the notion of *no-regret* develloped in the field of online convex optimization, proved to be a very meaningful benchmark in order to formally define natural behaviors for selfish agents in imperfect information settings.

2.5.1 Random-Payoff Opinion Formation Games

As already mentioned both the FJ model and its respective opinion formation game were very influential in modeling the opinion formation process. However there are notable cases in which the FJ model does not appropriately describe the dynamics of the opinions, due to the large amount of information exchange that it implies. More precisely, at each round $t \geq 1$ its update rule

$$x_i(t) = \frac{\sum_{j\neq i} w_{ij}x_j(t-1) + w_i s_i}{\sum_{j\neq i} w_{ij} + w_i}$$

requires that every agent learns *all* the opinions the agents with $w_{ij} > 0$! In today's large social networks where users usually have several hundreds of friends it is highly unlikely that, each day they learn the opinions of all their social neighbors. In such environments it is far more reasonable to assume that individuals randomly meet a small subset of their acquaintances and these are the only opinions that they learn.

In order to capture the above motivation, we consider a *random payoff variant* of the opinion formation game introduced in [17]. For a given opinion vector $x = (x_i, x_{-i}) \in [0,1]^n$, the disagreement cost of agent i is the following random variable $C_i(x_i, x_{-i})$:

- Agent i meets **just one** of her neighbors j with probability,

$$p_{ij} = \frac{w_{ij}}{\sum_{j \neq i} w_{ij}}$$

- Agent i suffers cost

$$C_i(x_i, x_{-i}) = (1 - \alpha_i)(x_i - x_j)^2 + \alpha_i (x_i - s_i)^2,$$

  where $\alpha_i = w_i / (\sum_{j \neq i} w_{ij} + w_i)$.

This random payoff variant is based on the natural assumption that the influence between two individuals in a society is the frequency that these individuals interact. It is not hard to see that for a given opinion vector $(x_i, x_{-i}) \in [0,1]^n$ the expected disagreement cost of agent i is proportional to the disagreement cost of Equation 2.1 (i.e. $\mathbf{E}\left[C_i(x_i, x_{-i})\right] \sim \sum_{j \neq i} w_{ij}(x_i - x_j)^2 + w_i(x_i - s_i)^2$). As a result, the Nash Equilibrium of this random-payoff opinion formation game (defined with respect to the expected disagreement cost) is the same with the Nash Equilibrium of the opinion formation game defined in [17]. Moreover the *simultaneous best response dynamics* (with respect to the expected disagreement cost) is an instance of the FJ model.

Contribution

We study the convergence properties of *natural and efficient dynamics* in this random payoff opinion formation game. By the term *natural* we mean that the agents update their opinions in their effort to minimize their disagreement cost. By the term *efficient* we mean that the update rule of the *dynamics* respect the information exchange constraints of the game: at every round each agent learns *just* the opinion of the agent that she randomly met.

Example 2.2.

- *The FJ model is natural: each agent selects the opinion that minimizes her expected disagreement cost with respect to the expressed opinions of her neighbors in the previous round.*

- *The FJ model is not efficient: in order to compute her best-response opinion, agent i must know the opinions of **all** the agents with $w_{ij} > 0$. Thus, this update rule does not respect the information exchange constraints of random-payoff opinion formation game.*

Although the term *efficient* is very clear (learning just the opinion of the randomly-met agent), the term *natural* is totally ambigous in this limited information exchange setting. Since each selfish agent learns the opinion of *just* one of her neighbors at the end of each round, it is not clear at all what is *natural* for such an agent to do in her attempt to to minimize her individual disagreement cost. The *online convex optimization* framework provides a very concrete answer to what the agent can do in this limited information setting and clarifies the word *natural dynamics*.

An agent can update her opinion so as the disagreement cost that she experiences is smaller than the disagreement cost that she would experience by expressing any fixed opinion. This will hold no matter the opinions of the randomly-met neighbors.

As already mentioned, the latter guarantee is refered as *no-regret* in the *online convex optimization* literature and the existence of such *no-regret* algorithms had a vast influence on online decision making (see also [87] for an introduction to online convex optimization).

In Chapter 3, we present a limited-information exchange variant of the FJ model, called *Follow the Leader dynamics* which is both natural and efficient in the above-presented sense. Its update rule requires only the opinion of the randomly-met-agent, making it efficient. At the same time when an agent uses this rule to update her opinion, she is ensured *no-regret* to her experienced disagreement cost even if the opinions of the randomly-met agents where selected by a malicious adversary. As already discussed, the no-regret guarantee makes our rule a natural choice for selfish agents that are only interested in their individual disagreement cost. We show that if this update rule is adopted by all agents, then the produced opinion dynamics (*Follow the Leader dynamics*) is ε-close to Nash Equilibrium in $\tilde{O}(1/\varepsilon^2)$ rounds. We also remark that this rule is fairly simple (roughly a time-average on the observed opinions) and it is based on the Follow the Regularized Leader algorithm develloped for online convex optimization problems [87]. Moreover *Follow the Leader dynamics* comes as a very simple and intuitive limited-information exchange variant of the original FJ model and its convergence property adds robustness to the predictive power of the Nash Equilibrium $x^* \in [0,1]^n$ of the FJ model.

We then ask whether there exists an update rule that ensures *no-regret* and produces opinion dynamics with faster convergence rate. Motivated by this question we discover a very interesting phenomena: *every dynamics that is at the same time both natural and efficient, needs at least* $\Omega(1/\varepsilon)$ *rounds to be* ε*-close to Nash Equilibrium. However this is not true for dynamics that* *are just efficient*. We prove this lower bound on the convergence rate of such dynamics through the use of an information-theoretic argument that connects no-regret dynamics with the statistical estimation of the success probability of a Bernoulli random variable.

We finally seek for update rules producing opinion dynamics that are just *efficient* and that converge exponentially fast to Nash Equilibrium x^*. We remark that the existence of such dynamics is not excluded by the above lower bound. Combining ideas from recent stochastic gradient descent methods [94, 131] and from older distributed gradient descent methods [13], we design an update rule that does not ensure the no-regret property (the produced dynamics is not natural), but the produced dynamics is ε-close to Nash Equilibrium in $O(\log^2(1/\varepsilon))$ rounds. The key idea is that learning the opinion of *just one* randomly selected neighbor, can be seen as having access to an oracle producing a random vector with expected value equal to the gradient of the potential function $\Phi(x_i, x_{-i})$ of Equation 2.2. Our dynamics can be seen as a distributed protocol that appropriately uses this «noisy gradient» to minimize the potential function $\Phi(x_i, x_{-i})$ in as few rounds as possible.

2.5.2 Opinion Formation Games with Aggregation and Negative Influence

In many domains, public opinions are not only affected by local interactions and personal beliefs, but also by influences that stem from global properties of the opinions present in the society. People are getting exposed to global trends, societal norms, results from voting and polling, etc., which are usually interpreted as the consensus view of the society and may crucially affect opinion formation. Furthermore, groups of people (or networks of agents) often need to agree on a common action, even if their beliefs and/or their expressed opinions are totally different. This might happen e.g., when some network devices need to implement a common action, when people vote over a set of alternatives, or when a wisdom-of-the-crowd opinion is formed in a social network.

We capture such situations by assuming that an *aggregation rule* maps the public opinions to a *global* opinion that represents the consensus view on the issue at hand. The agents anticipate the impact of their public opinions on the global one and might incorporate it in their opinion selection. This means that the disagreement cost should also account for the distance of an agent's intrinsic belief to the global opinion. To address these issues, we consider a generalization of the opinion formation game of [17] with opinion aggregation. The strategy of each agent is her public opinion $x_i \in \mathbf{R}$, while for a given public opinions vector $x = (x_i, x_{-i}) \in \mathbf{R}^n$ agent's i cost is

$$C_i(x) = \sum_{j \neq i} w_{ij}(x_i - x_j)^2 + w_i(x_i - s_i)^2 + \alpha_i(aggr(x) - s_i)^2$$

where $aggr : \mathbf{R}^n \mapsto \mathbf{R}$ maps the public opinion vector to an aggregated global opinion and the weight $\alpha_i \geq 0$ quantifies the appeal of this global opinion $aggr(x)$ to agent i. As in the opinion formation game defined in [17], $w_{ij} \geq 0$ denotes the influence of that agent j poses on agent i and $s_i \in [0, 1]$ the internal opinion of agent i.

Motivated by previous work on the wisdom of the crowd [92, Sec. 8.3], [83], we concentrate on *average-oriented opinion formation games*, where the aggregation rule $aggr(x)$ is the average public opinion

$$avg(x) = \sum_{j=1}^{n} x_j/n$$

Contribution

The opinion formation game introduced in [17] is special case of the *average-oriented opinion formation game* where all coefficients $\alpha_i = 0$. At a first glance, the additional aggregation term seems not to have a major impact on the properties of the game. This is far from being true! As we shall see in Chapter 4, the presence of this simple aggregation term introduces negative influence among the agents and this crucially affects both the PoA bounds and the convergence properties of the *simultaneous best response dynamics* (FJ model). The following example reveals that in *average-oriented opinion formation game* even the existence of Nash Equilibrium is not guaranteed.

Example 2.3. *Let the two-player average-oriented opinion formation game with*

- $w_{12} = w_{21} = 0$
- $w_1 = w_2 = 0$
- $\alpha_1 = \alpha_2 = 1$
- $s_1 = 0$ *and* $s_2 = 1$

In this instance $C_1(x_1, x_2) = (\frac{x_1+x_2}{2})^2$ *and* $C_2(x_1, x_2) = (\frac{x_1+x_2}{2} - 1)^2$. *Thus* $\frac{dC_1(x_1,x_2)}{dx_1} = \frac{x_1+x_2}{2}$ *and* $\frac{dC_2(x_1,x_2)}{dx_2} = \frac{x_1+x_2}{2} - 1$. *Meaning that there is no vector* $(x_1^*, x_2^*) \in \mathbf{R}^2$ *such as* $\frac{dC_1(x_1^*,x_2^*)}{dx_1} = 0$ *and* $\frac{dC_2(x_1^*,x_2^*)}{dx_2} = 0$ *at the same time. Since* *the strategy space of the agents is* $\mathbf{R}$*, at any Nash Equilibrium,* $\frac{dC_1(x_1^*,x_2^*)}{dx_1} = 0$ *and* $\frac{dC_2(x_1^*,x_2^*)}{dx_2} = 0$*. The latter implies that there is no Nash Equilibrium for the above instance of the game.*

As Example 2.3 illustrates, there are instances of the average-oriented opinion formation game in which *simultaneous best-response dynamics* does not converge since Nash Equilibrium does not even exist. Thus we examine under which circumstances the nice properties of the FJ model are restored. We provide general and intuitive conditions about the coefficients w_i, α_i (see Assumption 1 in Chapter 4) under which not only the existence of Nash Equilibrium is guaranteed, but also *simultaneous best response dynamics* converges fast to it. To provide the high level idea on how these conditions are derived, we remark that when the game is *symmetric* i.e. $w_{ij} = w_{ji}$ and $\alpha_i = \alpha$, the function

$$\Phi(x_i, x_{-i}) = \sum_{j \neq i} w_{ij}(x_i - x_j)^2 + \sum_{j \in V} w_j(x_j - s_j)^2 + \alpha(\sum_{j \in V} \frac{x_j}{n})^2 - 2\alpha \sum_{j \in V} \frac{s_j x_j}{n} \tag{2.3}$$

serves as a *potential function* of the game, $\Phi(x_i, x_{-i}) - \Phi(x_i', x_{-i}) = C_i(x_i, x_{-i}) - C_i(x_i', x_{-i})$. The latter implies that any local minimum of $\Phi(x_i, x_{-i})$ (if such exists) is a Nash Equilibrium of the game and vice versa. Moreover the *simulataneous best response dynamics* corresponds to a step of the Newton method in $\Phi(x_i, x_{-i})$. Roughly speaking, the role of the conditions stated in Assumption 1 is to is to make $\Phi(x_i, x_{-i})$ convex so as the Newton method converges to its unique minimizer. We highly remark that Assumption 1 does not require neither $w_{ij} = w_{ji}$ nor $\alpha_i = \alpha$ for all agents i, meaning that *simultaneous best-response dynamics* converges even if the potential function of Equation 2.3 does not exist. Our first convergence result states that under Assumption 1, *simultaneous best-response dynamics* is ε-close to Nash Equilibrium in $O(n^2 \log n/\varepsilon)$ rounds.

The *simultaneous best response dynamics* in the *average-oriented opinion formation games* implies even larger amount of information exchange than the original FJ model. Now an agent must learn at each round not only the opinions of her friends ($w_{ij} > 0$), but also the *average public opinion* in order to compute her best response. Obviously the *average global opinion* is an expensive information to obtain in realistic settings and thus we examine an *outdated* version of the *simultaneous best response dynamics.* Now agents learn the opinions of their social neighbors at each round, but the average public opinion is announced to them every now and then. At each round, the agents compute their best response opinion with respect to the opinions of her friends in the previous round and the most recently announced average public opinion. We show that similar convergence results hold even with this apparent reduction in the information exchange. More precisely we show that *outdated simultaneous best-response dynamics* is ε-close to Nash Equilibrium after $O(n^2 \log n/\varepsilon)$ announcements of the *average public opinion*. This result in based on techniques develloped in the context of distributed gradient descent methods [13, 41]. In the distributed convex optimization framework, n processors try to minimize a convex function of n variables while each processors is responsible for a specific variable of the function. The basic problem is that a processor may have outdated information about the value of some variables due to intermediate updates of other processors. The major difficulty in proving that *outdated simultaneous best-response dynamics* converges to Nash Equilibrium despite the fact that agents have outdated information about the *average public opinion*, is that the conditions stated in Assumption 1 do not imply the existence of a potential function of Equation 2.3 and thus reaching Nash Equilibrium is not equivalent with minimizing a convex potential function.

The crucial difference between the opinion formation game introduced in [17] and our average-oriented opinion formation game is that in the second case there are equilibria in which some agents adopt opinions outside the $[0, 1]$ interval ($[0, 1]$ is the interval in which the internal opinions s_i lie). The latter is an effect of the negative influence among the agents introduced by the additional averaging term. Although assuming that the agents can select any opinion in the real line is not by principle a bad assumption, there are settings such as voting in which opinions must necessarily lie in a fixed range. To cover such settings, we consider the *restricted version* of the game in which the strategy space of the agents is the $[0, 1]$ interval (and not the entire $\mathbf{R}$). We prove that under the conditions of Assumption 1, both the *simultaneous best response dynamics* and its *outdated variant* converge to Nash Equilibrium of the *restricted version* of the game.

Finally we turn our attention to the quality of Nash equilibrium in terms of total disagreement cost. Using the Local Smoothness technique introduced in [128], we show that in the *unrestricted version* of the game the *Price of Anarchy* is $9/8 + O(\alpha/(wn^2))$ if $w_i = w$ and $\alpha_i = \alpha$ for all agents i. For the *restricted version*, which is much harder to analyze, we show that the *Price of Anarchy* is $3 + \sqrt{2}$ if $w_i = \alpha_i = 1$ for all agents i.

2.5.3 Network Hegselmann Krause Model

As already discussed, the HK model had a vast influence on the study of opinion formation. Howevever the model implicitly assumes something rather questionable. Whenever two agents i, j share similar opinions $|x_i(t) - x_j(t)| \leq$

ε, then there are mutually influenced. Apart from having similar opinions individuals must be in sense socially connected in order to be influenced, at least they must know each other!

We introduce a very straightforward generalization of the HK model to capture the above issues. We assume the existence of an undirected graph $G = (V, E)$ in which V stands for the agents and E for the social relations among them. In our generalization called Network HK model, each agent averages her current opinion with the opinions of her neighbors in G that are ε-close to hers.

Network Hegselmann Krause model

1: An undirected graph $G = (V, E)$, $|V| = n$
2: $x_i(0) \in [0, 1]$, agent's i initial opinion.
3: At round $t \geq 1$, each agent i updates her opinion:

$$x_i(t) = \frac{\sum_{j \in N_i(t)} x_j(t-1) + x_i(t-1)}{|N_i(t)| + 1}$$

where $N_i(t) = \{j \neq i : \ |x_i(t-1) - x_j(t-1)| \leq \epsilon \text{ and } (i,j) \in E\}$

The HK model is a special case of Network HK model when G is a clique. Similarly with HK model, the Network HK model admits an infinite number of stable points no matter the topology of G. Unfortunately the convergence results of the HK model cannot be easily generalized. The proof of convergence of the HK model heavily relies on the following fact: *the ordering of the agents from left to right according to their opinions is always the same!* [14, 19] This nice structure holds only when G is a clique and thus we devellop different techniques to analyze the convergence properties of the Network HK model.

Contribution

In Chapter 5, we show that Network HK model always converges to a stable state. We consider the sequence of undirected graphs that represent the influences among the agents at each round, G' s edges whose endpoints have opinions that are ε-close. We then use the notion of weak connectivity, introduced in [96], to prove that either the overall dynamical systems splits into independent sub-systems or all the agents adopt the same opinion. Some of our results coincide with results in [91, 105] concerning products of stochastic matrices, however our approach provides simpler and more versatile proofs.

2.5.4 Random Hegselmann Krause Model

In Section 2.5.1 we discussed about the large information exchange that the FJ model requires, rendering it unsuitable for modeling the opinion formation process in large social network. Things are much worse in the HK model from this point of view. Now agents need to learn *all* the opinions in order to determine which of the agents are in distance ε. We capture these issues with a straightforward variant called Random HK model: Each agent randomly meets k other agents and averages her opinion with those opinions that are ε-close to hers.

Random Hegselmann Krause model

1: n agents.
2: $x_i(0) \in [0, 1]$, agent's i initial opinion.
3: At round $t \geq 1$, each agent i:
    4: selects k other agents uniformly at random, $R_i(t) \subseteq [n]$.
    5: updates her opinion,

$$x_i(t) = \frac{\sum_{j \in N_i(t)} x_j(t-1) + x_i(t-1)}{|N_i(t)| + 1}$$

where $N_i(t) = \{j \neq i : \ |x_i(t-1) - x_j(t-1)| \leq \varepsilon \text{ and } j \in R_i(t)\}$

Contribution

In Chapter 5, we show that Random HK model always reaches a stable state. As in Network HK model, the ordering of the agents (according to their opinions) is not preserved and as a result the techniques for proving convergence in the original HK model do not apply. Random HK model involves an additional difficulty, since it employs asymmetric influence between the agents (it may be an agent i influences agent j, while j does not influence i) and averaging systems that permit directionality are notoriously difficult to be analyzed [37, 15]. As in the case of Network HK model, we use a suitably adjusted notion of weak connectivity to prove that with high probability either the system splits into independent sub-systems or all the agents adopt the same opinion.

2.5.5 Reallocating Facilities on the Line

In Chapter 6, we present our results concerning a *dynamic version* of the well-studied *K-median problem* called *K-facility reallocation problem* that was introduced in [58].

In the K*-facility reallocation*, K facilities are initially located at $(x_1^0, \ldots, x_K^0)$ on the real line. Facilities are meant to serve n agents for the next T days. At each day each agent connects to the facility closest to its location, incuring her a connection cost equal to their distance. The locations of the agents may change from day to day and thus facilities have to accordingly move in order to reduce the connection cost. Naturally, moving a facility is not for free, but comes with the price of the distance that the facility was moved. Our goal is to specify the exact positions of the facilities at each day so that the total connection cost plus the total moving cost is minimized over all T days.

A very motivating application of the K*-facility reallocation* comes from the world of opinion selection. Assume that a political with K candidates wants to win the next T elections. The opinions of the voters are points of the real time that may change from time to time. A party would like to adjust the expressed opinions of its candidates so as to represent as many voters as possible. A voter is more likely to vote for the party at the t-th elections if is at least one of its representatives expresses an opinion similar to the voter's opinion at that time. However politicians with constantly changing opinions may not be taken seriously by the public and this should be taken into account by the party. As a result, the party should assign opinions to its candidates so as to efficiently cover the political spectrum and at the same time its candidates do not dramatically change their public positions.

Contribution

We resolve the computational complexity of K-*facility reallocation problem* on the real line. In Capter 6, we present an optimal algorithm with running time polynomial in the parameters n, T and K. This substantially improves on the complexity of the algorithm, presented in [58], that is exponential in K. Our algorithm solves a Linear Programming relaxation and then *rounds* the *fractional solution* to determine the positions of the facilities. Our main technical contribution consists in showing that a simple rounding scheme yields an integral solution with the exact same cost as the *fractional one*.

Related work

We can cast the K*-facility reallocation problem* as a clustering problem on a temporally evolving metric. From this point of view, K*-facility reallocation problem* is a dynamic K*-median* problem. A closely related problem is the *dynamic facility location problem*, [63, 3]. Other examples in this setting are the *dynamic sum radii clustering* [18] and multi-stage optimization problems on matroids and graphs [84]. In [80], a mobile facility location problem was introduced, which can be seen as a one stage version of our problem. They showed that even this version of the problem is NP-hard in general metric spaces using an approximation preserving reduction to K-*median problem*.

Online facility location problems and variants have been extensively studied in the literature, see [74] for a survey. [61] studied an online model, where facilities can be moved with zero cost. As we have mentioned before, the online variant of the K-*facility reallocation problem* is a generalization of the K-*server problem*, which is one of the most natural online problems. [100] showed a $(\mathbf{2}K - \mathbf{1})$-competitive algorithm for the K-*server problem* for every metric space, which is also K-competitive, in case the metric is the real line [11]. Other variants of the K-*server problem* include the (H, K)-*server problem* [9, 8], the *infinite server problem* [51] and the K-*taxi problem* [71, 50].

Chapter 3

Random-Payoff Opinion Formation Games

In this chapter we present our results on random-payoff opinion formation games. We introduced this kind of games in our work [75] in order to capture the fact that people form opinions by just learning a small number of their social circle. Such issues have not been considered in the original FJ model [79] and its respective opinion formation game [17].

We are interested in the convergence properties of simple and natural variants of the FJ model that use limited information exchange. More precisely, each agent learns *just one* opinion of the other agents at each round. To address these questions, one could define precise dynamical processes whose update rules satisfy these information exchange requirements and study their convergence properties. But what is the modelling power of such processes? How can we formally define what natural means in order to rule out complex algorithmic distributed protocols that certainly have nothing to do with the way people form opinions?

Instead of proposing ad-hoc models that resemble the FJ model to a bigger or a lesser extend we adopt a more structured approach. We introduce a random-payoff variant of the opinion formation game defined in [17], capturing the fact that each agent meets *just one* other agent, and we assume that agents iteratively play this game. This way we can define as natural variants of FJ, update rules that minimize in some sense the disagreement cost of the agents and to study general classes of dynamics (e.g. no-regret dynamics) without explicitly defining their update rule.

3.1 Random-Payoff Opinion Formation Games

According to the opinion formation game of Bindel, Kleinberg and Oren [17], each agent i expresses an opinion x_i so as to minimize her disagreement cost

$$C_i(x_i, x_{-i}) = \sum_{j \neq i} w_{ij}(x_i - x_j)^2 + w_i(x_i - s_i)^2 \tag{3.1}$$

Since the FJ model is the best response dynamics of this game, this opinion formation game inherits all of its modelling and predictive power on the way opinions are formed. However at a first glance something fairly unreasonable seems to be introduced: each agent i somehow interacts with the all the agents with $w_{ij} > 0$ in order to experience this disagreement cost. But how this is done if this number is of several hundreds for each agent?

Random-payoff games provide a simple and intuitive fix to the above critism. An agent i randomly meets *just one* of her friends and the weight w_{ij} describe the probability of meeting her friend j. Now Equation 3.1 can be interpreted as the expected disagreement cost that an agent experiences. This comes along with the general belief that we are influenced more by those we interact more often. The above discussion is formally captured in the random-payoff game of Definiton 3.1. In these games the disagreement cost of agent i for expressing the opinion x_i is random variable $C_i(x_i, x_{-i})$ whose expected value is given by Equation 3.1.

Definition 3.1. *For a given opinion vector $x = (x_i, x_{-i})$ the cost of each agent i is the random variable $C_i(x_i, x_{-i})$ defined as follows:*

- *Each agent i randomly meets* just one *agent W_i,*

$$\mathbf{P}\left[W_i = j\right] = \frac{w_{ij}}{\sum_{j \neq i} w_{ij}}$$

- *Experiences disagreement*

$$C_i(x_i, x_{-i}) = (1 - \alpha_i)(x_i - x_{W_i})^2 + \alpha_i(x_i - s_i)^2$$

where $\alpha_i = w_i / (\sum_{j \neq i} w_{ij} + w_i)$.

The random-payoff game of Definition 3.1 introduces a higher of abstraction. Both the FJ model and the game in [17] (see Equation 3.1) can be obtained by assuming that the agents are interested in minimizing the expected disagreement cost. More precisely consider that the agents iteratively play the random-payoff game of Definition 3.1. Let's assume that at the end of each round somehow each agent is informed about the opinions of the others. If the agents update their public opinion so as to minimize their expected disagreement in the next round based on the opinions that they learned, then FJ model comes out! At the same time let the opinions of the agents be the unique Nash Equilibrium x^* of the original opinion formation. Let us assume that the agents are able to change their opinions before the random meetings take place. Then nobody change her opinion because this would increase her expected disagreement cost during the random meetings. This also reveals that the Nash Equilibrium x^* of the game in [17] (stable point of FJ model) is also a meaningful notion in our random-payoff opinion formation games.

Definition 3.2. *An opinion $x^* = (x_i^*, x_{-i}^*) \in [0,1]^n$ is a Nash Equilibrium if and only if for each agent i*

$$\mathbf{E}\left[C_i(x_i^*, x_{-i}^*)\right] \geq \mathbf{E}\left[C_i(x_i, x_{-i}^*)\right] \text{ for every } x_i \in [0,1]$$

Random-payooff games provides us with a holistic framework for studying the opinion formation proccess under the modelling principles of Friedkin and Johnsen. The agents are assumed to repeatedly play the game of Definition 3.1 and at the end of each round they update their opinions so as to minimize their disagreement cost. The exact way this updating is performed and the exact assumption on what the agents know about the opinions of the others, leads to different opinion dynamics however all the them respect the modelling principles that Friendkin and Johnsen initially posed. Obviously the most natural thing to consider is that the agents learn at each round only the opinion of the randomly-met-agent and then use this information to minimize their disagreement cost, but we highly remark that this framework does not prohibit someone to consider different information exchange assumptions.

Throughout this chapter, we consider that at each round each agent *learns only* the opinion of her randomly-met-agent. But now the following question arises.

Question 1. *What is reasonable for the agents to do with such little information in order to minimize their disagreement cost?*

Clearly the natural choice of best response (that FJ model assumes) is no longer an option since the agents do not know enough in order to compute it. However there is something much more reasonable than best response that the agents can do. They can select their opinions according to a *no-regret* algorithm for the following online convex optimization problem:

Definition 3.3. At round $t \geq 0$, *1.* the agent selects a value $x_t \in [0, 1]$.

2. the adversary observes the x_t and selects a $b_t \in [0, 1]$

3. the agent receives cost $(1 - \alpha_i)(x_t - b_t)^2 + \alpha_i(x_t - s_i)^2$.

where $b_t \in [0, 1]$

Each agent i is very eager to update her opinion according to such a no-regret algorithm because such algorithms guarantee that the disagreement cost that the agent experiences during the game play is close to the disagreement cost that she would experience by selecting the *best fixed opinion* during the whole game play. In Section 3.4 we formally present the no-regret guarantees and a brief introduction to the online convex optimization framework.

Our work studies the opinion dynamics when the agents update their opinion according to such no-regret algorithms. More precisely we shed light on the following questions:

Question 2.
- *What is the limiting behavior the opinions if such algorithms are adopted by the agents?*
- *Are there simple update rules such that*
  - *no-regret is ensured to any agent that adopts them.*
  - *the overall system converges to the Nash Equilibrium* x^*.

We present a very simple and intuitive update rule that meets the requirements of Question 2. It ensures no-regret to any agent that adopts and the same time the produced dynamics converges to Nash Equilibrium x^*. These results are formally stated and proven in Theorem 3.2 and Theorem 3.1 respectively. We then prove that any opinion dynamics produced by update rules that ensure no-regret to the agents cannot have much faster convergence rate, whereas we find an update rule that does not ensure no-regret to the agents, but its produced dynamics converge to the Nash Equilibrium of the game exponentially fast.

3.2 Our Results

Before presenting our results we introduce some necessary notation. For simplicity we adopt the following notation for an instance of the game of Definition 3.1.

Definition 3.4. *We denote an instance of the opinion formation game of Definition 3.1 as triple $I = (P, s, \alpha)$, where*

- *P is a $n \times n$ stochastic matrix.*
- *$s \in [0, 1]^n$ is the internal opinion vector.*
- *$\alpha \in (0, 1]^n$ the self confidence vector.*

Corollary 3.1. *For a given instance $I = (P, s, \alpha)$ the Nash equilibrium $x^* \in [0, 1]^n$ is the unique solution of the following linear system:*

$$x_i^* = (1 - \alpha_i) \sum_{j \neq i} p_{ij} x_j^* + a_i s_i, \text{ for every agent } i$$

.

The proof of Corollary 3.1 follows directly by the definition of Nash Equilibrium (Definition 3.2) though some simple algebra, while the fact that the above linear system always admits a solution follows by matrix norm properties. Throughout the chapter we study *dynamics* of the random-payoff game of Definition 3.1. We denote as W_i^t the neighbor that agent i met at round t, which is a random variable whose probability distribution is determined by the instance $I = (P, s, \alpha)$ of the game, $\mathbf{P}[W_i^t = j] = p_{ij}$. Another parameter of an instance I that we often use is $\rho = \min_{i \in V} \alpha_i$.

In Section 3.3, we examine the convergence properties of the opinion vector $x(t)$ when all agents update their opinions according to the *Follow the Leader* principle. Since each agent i must select $x_i(t)$, before knowing which of her neighbors she will meet and what opinion her neighbor will express, this update rule says «*play the best according to what you have observed*». The convergence rate of Follow the Leader dynamics to the unique Nash Equilibrium x^* is stated and proven in Theorem 3.1.

Follow the Leader dynamics

Initially $x_i(0) = s_i$ for all agents i.
At round $t \geq 0$ each agent i:
Meets neighbor with index W_i^t where $\mathbf{P}\left[W_i^t = j\right] = p_{ij}$
Suffers cost disagreement cost

$$(1-\alpha_i)(x_i(t) - x_{W_i^t}(t))^2 + a_i(x_i(t) - s_i)^2$$

and learns the opinion $x_{W_i^t}(t)$.
Updates her opinion as follows

$$x_i(t+1) = \underset{x \in [0,1]}{\operatorname{argmin}} \sum_{\tau=0}^{t} (1-\alpha_i)(x - x_{W_i^\tau}(\tau))^2 + \alpha_i(x - s_i)^2 \tag{3.2}$$

Theorem 3.1. *Let $I = (P, s, \alpha)$ be an instance of the opinion formation game of Definition 3.1 with equilibrium $x^* \in [0,1]^n$. The opinion vector $x(t) \in [0,1]^n$ produced by update rule (3.2) after t rounds satisfies*

$$\mathbf{E}\left[\|x(t) - x^*\|_\infty\right] \leq C\sqrt{\log n}\frac{(\log t)^{3/2}}{t^{\min(1/2,\rho)}},$$

where $\rho = \min_{i \in V} a_i$ and C is a universal constant.

In Section 3.4 we argue that, apart from its simplicity, update rule (3.2) ensures no-regret to any agent that adopts it and therefore the FTL dynamics can be considered as *natural dynamics* for selfish agents. Since each agent i selfishly wants to minimize the disagreement cost that she experiences, it is natural to assume that she selects $x_i(t)$ according to a *no-regret algorithm* for the *online convex optimization problem* where the adversary chooses a function $f_t(x) = (1-\alpha_i)(x-b_t)^2 + \alpha_i(x-s_i)^2$ at each round t. In Theorem 3.2 we prove that *Follow the Leader* is a no-regret algorithm for the above OCO problem. We remark that this does not hold, if the adversary can pick functions from a different class (see e.g. chapter 5 in [87]).

Theorem 3.2. *Consider an arbitrary sequence $(b_t)_{t=0}^\infty$ and the function $f : [0,1]^2 \mapsto [0,1]$ with $f(x,b) = (1-\alpha)(x-b)^2 + \alpha(x-s)^2$ for some constants $s, \alpha \in [0,1]$. Then for all $t \geq 0$,*

$$\frac{1}{t}\sum_{\tau=0}^{t} f(x_\tau, b_\tau) \leq \frac{1}{t}\min_{x \in [0,1]}\sum_{\tau=0}^{t} f(x, b_\tau) + O\left(\frac{\log t}{t}\right).$$

where $x_t = \operatorname{argmin}_{x \in [0,1]} \sum_{\tau=0}^{t-1} f(x, b_\tau)$

On the positive side, the FTL dynamics converges to x^* and its update rule is simple and ensures no-regret to the agents. On the negative side, its convergence rate is outperformed by the rate of FJ model. For a fixed instance $I = (P, s, \alpha)$, the FTL dynamics converges with rate $\widetilde{O}(1/t^{\min(\rho,1/2)})$ while FJ model converges with rate $O(e^{-\rho t})$ [81].

Question 3. *Can the agents adopt other no-regret update rules such that the resulting dynamics converges fast to x^*?*

The answer is no. In Section 3.5, we prove that fast convergence cannot be established for any *no-regret dynamics*. The reason that FTL dynamics converges slowly is that rule (3.2) only depends on the opinions of the neighbors that agent i meets, α_i, and s_i. This is also true for any update rule that ensures no-regret to the agents (see Section 3.5). We call the larger class of update rules that do not use the values p_{ij} *graph oblivious* (this class includes all the no-regret algorithms) and we prove that fast convergence cannot be established for any *graph oblivious dynamics*.

Definition 3.5 (graph oblivious update rule)**.** *A graph oblivious update rule A is a sequence of functions $(A_t)_{t=0}^{\infty}$ where $A_t : [0,1]^{t+2} \mapsto [0,1]$.*

Definition 3.6 (graph oblivious dynamics)**.** *Let a graph oblivious update rule A. For a given instance $I = (P, s, \alpha)$ the rule A produces a graph oblivious dynamics $x_A(t)$ defined as follows:*

- *Initially each agent i selects her opinion $x_i^A(0) = A_0(s_i, \alpha_i)$*
- *At round $t \geq 1$, each agent i selects her opinion*

$$x_i^A(t) = A_t(x_{W_i^0}(0), \ldots, x_{W_i^{t-1}}(t-1), s_i, \alpha_i)$$

  where W_i^t is the neighbors that i meets at round t.

Note that FTL dynamics is a graph oblivious dynamics since update rule (3.2) can be written equivalently, $x_i(t) = (1-\alpha_i)\sum_{\tau=0}^{t-1} x_{W_i^\tau}(\tau)/t + \alpha_i s_i$. Theorem 3.3 states that for any graph oblivious dynamics there exists an instance $I = (P, s, \alpha)$, where roughly $\Omega(1/\varepsilon)$ rounds are required to achieve convergence within error ε.

Theorem 3.3. *Let A be a* graph oblivious *update rule, which all agents use to update their opinions. For any $c > 0$ there exists an instance $I = (P, s, a)$ such that*

$$\mathbf{E}\left[\|x_A(t) - x^*\|_\infty\right] = \Omega(1/t^{1+c}),$$

where $x_A(t)$ denotes the opinion vector produced by A for the instance $I = (P, s, \alpha)$.

To prove Theorem 3.3, we show that graph oblivious rules whose dynamics converge fast imply the existence of estimators for Bernoulli distributions with «small» sample complexity. The key part of the proof lies in Lemma 3.6, in which it is proven that such estimators cannot exist. We also briefly discuss two well-known sample complexity lower bounds from the statistics literature and explain why they do not work in our case.

In Section 3.6, we present a simple update rule that achieves error rate $e^{-\widetilde{O}(\sqrt{t})}$. This update rule is a function of the opinions and the indices of the neighbors that i met, s_i, α_i and the i-th row of the matrix P. Obviously this rule is not *graph oblivious*, due to its dependency on the i-th row and the indices, and thus does not ensure no-regret to an agent that adopts it (see Example 3.1 in Section 3.6). However it reveals that slow convergence is not a generic property of the limited information dynamics, but comes with the assumption that agents act selfishly.

3.3 Convergence Rate of FTL Dynamics

In this section we prove Theorem 3.1 which bounds the convergence time of FTL dynamics to the unique equilibrium point x^*. At first notice that the update rule (3.2) of FTL dynamics can be equivalently written in the form of update rule (3.3).

Follow the Leader dynamics

1: Initially $x_i(0) = s_i$ for all agents i.
2: At round $t \geq 0$ each agent i:
   3: Meets neighbor with index W_i^t where $\mathbf{P}\left[W_i^t = j\right] = p_{ij}$
   4: Updates her opinion as follows

$$x_i(t) = (1 - \alpha_i)\frac{\sum_{\tau=0}^{t-1} x_{W_i^\tau}(\tau)}{t} + \alpha_i s_i \tag{3.3}$$

Since the opinion vector $x(t)$ is a random vector, the convergence metric used in Theorem 3.1 is $\mathbf{E}\left[\|x(t) - x^*\|_\infty\right]$ where the expectation is taken over the random meeting of the agents. The proof of Theorem 3.1 is quite technically complicated so we first present the high level idea. We remind that the unique equilibrium $x^* \in [0, 1]^n$ of the instance $I = (P, s, \alpha)$ satisfies the following equations for each agent $i \in V$,

$$x_i^* = (1 - \alpha_i)\sum_{j \neq i} p_{ij} x_j^* + \alpha_i s_i$$

Since our metric is $\mathbf{E}\left[\|x(t)-x^*\|_\infty\right]$, we can use the above equations to bound $|x_i(t)-x_i^*|$.

$$\begin{aligned} |x_i(t)-x_i^*| &= (1-\alpha_i)\left|\frac{\sum_{\tau=0}^{t-1} x_{W_i^\tau}(\tau)}{t} - \sum_{j\neq i} p_{ij}x_j^*\right| \\ &= (1-\alpha_i)\left|\sum_{j\neq i}\frac{\sum_{\tau=0}^{t-1}\mathbf{1}[W_i^\tau = j]x_j(\tau)}{t} - \sum_{j\neq i} p_{ij}x_j^*\right| \\ &\leq (1-\alpha_i)\sum_{j\neq i}\left|\frac{\sum_{\tau=0}^{t-1}\mathbf{1}[W_i^\tau = j]x_j(\tau)}{t} - p_{ij}x_j^*\right| \end{aligned}$$

Now assume that $|\frac{\sum_{\tau=0}^{t-1}\mathbf{1}[W_i^\tau=j]}{t} - p_{ij}| = 0$ for all $t \geq 1$, then with simple algebraic manipulations one can prove that $\|x(t)-x^*\|_\infty \leq e(t)$ where $e(t)$ satisfies the recursive equation $e(t) = (1-\rho)\frac{\sum_{\tau=0}^{t-1} e(\tau)}{t}$, where $\rho = \min a_i$. It follows that $\|x(t)-x^*\|_\infty \leq 1/t^\rho$ meaning that $x(t)$ converges to x^*. Obviously the latter assumption does not hold, however since W_i^τ are independent random variables with $\mathbf{P}\left[W_i^\tau = j\right] = p_{ij}$, $|\frac{\sum_{\tau=0}^{t-1}\mathbf{1}[W_i^\tau=j]}{t} - p_{ij}|$ tends to 0 with probability 1. In Lemma 3.1 we use this fact to obtain a similar recursive equation for $e(t)$ and then in Lemma 3.2 we upper bound its solution.

Lemma 3.1. *Let* $\delta(t) = \sqrt{\ln(\pi^2 n t^2/6p)/t}$ *and* $e(t)$ *the solution of the recursion with*

$$e(t) = \delta(t) + (1-\rho)\frac{\sum_{\tau=0}^{t-1} e(\tau)}{t}$$

where $e(0) = \|x(0)-x^*\|_\infty$*, and* $\rho = \min_{i\in V}\alpha_i$*. Then,*

$$\mathbf{P}\left[\text{for all } t\geq 1,\ \|x(t)-x^*\|_\infty \leq e(t)\right] \geq 1-p$$

Proof. At first we prove that with probability at least $1-p$, for all $t \geq 1$ and all agents i:

$$\left|\frac{\sum_{\tau=0}^{t-1} x^*_{W_i^\tau}}{t} - \sum_{j\neq i} p_{ij}x_j^*\right| \leq \sqrt{\frac{\log(\pi^2 n t^2/(6p))}{t}} := \delta(t). \tag{3.4}$$

Since W_i^τ are independent random variables with $\mathbf{P}\left[W_i^\tau = j\right] = p_{ij}$ and $\mathbf{E}\left[x^*_{W_i^\tau}\right] = \sum_{j\neq i} p_{ij}x_j^*$. By the Hoeffding's inequality we get

$$\mathbf{P}\left[\left|\frac{\sum_{\tau=0}^{t-1} x^*_{W_i^\tau}}{t} - \sum_{j\neq i} p_{ij}x_j^*\right| > \delta(t)\right] < 6p/(\pi^2 n t^2).$$

To bound the probability of error for all rounds $t \geq 1$ and all agents i, we apply the union bound

$$\sum_{t=1}^{\infty} \mathbf{P}\left[\max_i \left|\frac{\sum_{\tau=0}^{t-1} x^*_{W_i^\tau}}{t} - \sum_{j\neq i} p_{ij}x_j^*\right| > \delta(t)\right] \leq \sum_{t=1}^{\infty} \frac{6}{\pi^2}\frac{1}{t^2}\sum_{i=1}^{n}\frac{p}{n} = p$$

As a result with probability at least $1-p$ we have that inequality (3.4) holds for all $t \geq 1$ and all agents i. We now prove our claim by induction. Let $\|x(\tau) - x^*\|_\infty \leq e(\tau)$ for all $\tau \leq t-1$. Then

$$\begin{aligned}
x_i(t) &= (1-\alpha_i)\frac{\sum_{\tau=0}^{t-1} x_{W_i^\tau}(\tau)}{t} + \alpha_i s_i \\
&\leq (1-\alpha_i)\frac{\sum_{\tau=0}^{t-1} x^*_{W_i^\tau} + \sum_{\tau=0}^{t-1} e(\tau)}{t} + \alpha_i s_i && (3.5)\\
&\leq (1-\alpha_i)\left(\frac{\sum_{\tau=0}^{t-1} x^*_{W_i^\tau}}{t} + \frac{\sum_{\tau=0}^{t-1} e(\tau)}{t}\right) + \alpha_i s_i \\
&\leq (1-\alpha_i)\left(\sum_{j\neq i} p_{ij}x_j^* + \delta(t) + \frac{\sum_{\tau=0}^{t-1} e(\tau)}{t}\right) + \alpha_i s_i && (3.6)\\
&\leq x_i^* + \delta(t) + (1-\rho)\left(\frac{\sum_{\tau=0}^{t-1} e(\tau)}{t}\right)
\end{aligned}$$

We get (3.5) from the induction step and (3.6) from inequality (3.4). Similarly, we can prove that $x_i(t) \geq x_i^* - \delta(t) - (1-\rho)\frac{\sum_{\tau=0}^{t-1} e(\tau)}{t}$. As a result $\|x(t) - x^*\|_\infty \leq e(t)$ and the induction is complete. Therefore, we have that with probability at least $1-p$, $\|x(t) - x^*\|_\infty \leq e(t)$ for all $t \geq 1$.

□

Lemma 3.2. *Let $e(t)$ be a function satisfying the recursion*

$$e(t) = \delta(t) + (1-\rho)\sum_{\tau=0}^{t-1} e(\tau)/t \text{ and } e(0) = \|x(0) - x^*\|_\infty,$$

where $\delta(t) = \sqrt{\ln(Dt^{2.5})/t}$, $\delta(0) = 0$, and $D > \mathrm{e}^{2.5}$ is a positive constant. Then

$$e(t) \leq \sqrt{2\ln(D)}\frac{(\ln t)^{3/2}}{t^{\min(\rho,\,1/2)}}.$$

Proof. Observe that for all $t \geq 0$ the function $e(t)$ the following recursive relation

$$e(t+1) = e(t)\left(1 - \frac{\rho}{t+1}\right) + \delta(t+1) - \delta(t) + \frac{\delta(t)}{t+1} \tag{3.7}$$

For $t = 0$ we have that

$$e(1) = (1-\rho)e(0) + \delta(1) = (1-\rho)e(0) + \sqrt{\ln D} \tag{3.8}$$

Observe that for $D > \mathrm{e}^{2.5}$, $\delta(t)$ is decreasing for all $t \geq 1$. Therefore, $\delta(t+1) - \delta(t) + \frac{\delta(t)}{t+1} \leq \frac{\delta(t)}{t+1}$ and from equations (3.7) and (3.8) we get that for all $t \geq 0$

$$e(t+1) \leq e(t)\left(1 - \frac{\rho}{t+1}\right) + \frac{\sqrt{\ln(D(t+1)^2)}}{(t+1)^{3/2}} \leq e(t)\left(1 - \frac{\rho}{t+1}\right) + \frac{\sqrt{2\ln(D(t+1))}}{(t+1)^{3/2}}$$

Now let $g(t) = \frac{\sqrt{2\ln(Dt)}}{t^{3/2}}$ to obtain for all $t \geq 1$

$$\begin{aligned}
e(t) &\leq (1 - \frac{\rho}{t})e(t-1) + g(t) \\
&\leq (1 - \frac{\rho}{t})(1 - \frac{\rho}{t-1})e(t-2) + (1 - \frac{\rho}{t})g(t-1) + g(t) \\
&\leq (1 - \frac{\rho}{t})\cdots(1-\rho)e(0) + \sum_{\tau=1}^{t} g(\tau)\prod_{i=\tau+1}^{t}(1 - \frac{\rho}{i}) \\
&\leq \frac{e(0)}{t^\rho} + \sum_{\tau=1}^{t} g(\tau)e^{-\rho\sum_{i=\tau+1}^{t}\frac{1}{i}} \\
&\leq \frac{e(0)}{t^\rho} + \sum_{\tau=1}^{t} g(\tau)e^{-\rho(H_t - H_\tau)} \\
&\leq \frac{e(0)}{t^\rho} + e^{-\rho H_t}\sum_{\tau=1}^{t} g(\tau)e^{\rho H_\tau} \\
&\leq \frac{e(0)}{t^\rho} + \frac{\sqrt{2}}{t^\rho}\sum_{\tau=1}^{t} \tau^\rho \frac{\sqrt{\ln(D\tau)}}{\tau^{3/2}} \\
&\leq \frac{e(0)}{t^\rho} + \frac{\sqrt{2\ln D}}{t^\rho}\sum_{\tau=1}^{t} \frac{\sqrt{\ln \tau}}{\tau^{3/2-\rho}}
\end{aligned}$$

We observe that

$$\sum_{\tau=1}^{t} \frac{\sqrt{\ln \tau}}{\tau^{3/2-\rho}} \leq \int_{\tau=1}^{t} \frac{\sqrt{\ln \tau}}{\tau^{3/2-\rho}} d\tau \tag{3.9}$$

since, $\tau \mapsto \frac{\sqrt{\ln \tau}}{\tau^{3/2-\rho}}$ is a decreasing function of τ for all $\rho \in [0, 1]$.

- If $\rho \leq 1/2$ then

$$\int_{\tau=1}^{t} \tau^\rho \frac{\sqrt{\ln \tau}}{\tau^{3/2}} d\tau \leq \sqrt{\ln t}\int_{\tau=1}^{t} \frac{1}{\tau} d\tau = (\ln t)^{3/2}$$

- If $\rho > 1/2$ then

$$\begin{aligned}\int_{\tau=1}^{t} \tau^{\rho} \frac{\sqrt{\ln \tau}}{\tau^{3/2}} d\tau &= \int_{\tau=1}^{t} \tau^{\rho-1/2} \frac{\sqrt{\ln \tau}}{\tau} d\tau \\ &= \frac{2}{3} \int_{\tau=1}^{t} \tau^{\rho-1/2} ((\ln \tau)^{3/2})' d\tau \\ &= \frac{2}{3} t^{\rho-1/2} (\ln t)^{3/2} - (\rho - 1/2) \frac{2}{3} \int_{\tau=1}^{t} \tau^{\rho-3/2} (\ln \tau)^{3/2} d\tau \\ &\leq \frac{2}{3} t^{\rho-1/2} (\ln t)^{3/2}\end{aligned}$$

□

Now Theorem 3.1 follows by direct application of Lemma 3.2.

Theorem 3.1. *Let $I = (P, s, \alpha)$ be an instance of the opinion formation game of Definition 3.1 with equilibrium $x^* \in [0, 1]^n$. The opinion vector $x(t) \in [0, 1]^n$ produced by update rule (3.2) after t rounds satisfies*

$$\mathbf{E}\left[\|x(t) - x^*\|_\infty\right] \leq C \sqrt{\log n} \frac{(\log t)^{3/2}}{t^{\min(1/2, \rho)}},$$

where $\rho = \min_{i \in V} a_i$ and C is a universal constant.

Proof. By Lemma 3.1 we have that for all $t \geq 1$ and $p \in [0, 1]$,

$$\mathbf{P}\left[\|x(t) - x^*\|_\infty \leq e_p(t)\right] \geq 1 - p$$

where $e_p(t)$ is the solution of the recursion, $e_p(t) = \delta(t) + (1 - \rho) \frac{\sum_{\tau=0}^{t-1} e_p(\tau)}{t}$ with $\delta(t) = \sqrt{\frac{\log(\pi^2 n t^2/(6p))}{t}}$. Setting $p = \frac{1}{12\sqrt{t}}$ we have that

$$\mathbf{P}\left[\|x(t) - x^*\|_\infty \leq e(t)\right] \geq 1 - \frac{1}{12\sqrt{t}}$$

where $e(t)$ is the solution of the recursion $e(t) = \delta(t) + (1 - \rho) \frac{\sum_{\tau=0}^{t-1} e_p(\tau)}{t}$ with $\delta(t) = \sqrt{\frac{\log(2\pi^2 n t^{2.5})}{t}}$. Since $2\pi^2 \geq e^{2.5}$, Lemma 3.2 applies and $e(t) \leq C \sqrt{\log n} \frac{\log t^{3/2}}{t^{\min(\rho, 1/2)}}$ for some universal constant C. Finally,

$$\mathbf{E}\left[\|x(t) - x^*\|_\infty\right] \leq \frac{1}{12\sqrt{t}} + \left(1 - \frac{1}{12\sqrt{t}}\right) C \sqrt{\log n} \frac{(\log t)^{3/2}}{t^{\min(\rho, 1/2)}} \leq \left(C + \frac{1}{12}\right) \sqrt{\log n} \frac{(\log t)^{3/2}}{t^{\min(\rho, 1/2)}}$$

□

3.4 Follow the Leader Ensures No-Regret

In this section we provide rigorous definitions of *no-regret* algorithms and explain why update rule (3.2) ensures no-regret to any agent that repeatedly plays the game of Definition 3.1. Based on the disagreement cost that the agents experience, we consider an appropriate *online convex optimization* problem. This problem can be viewed as a «game» played between an adversary and a player. *At round* $t \geq 0$,

1. *the player selects a value* $x_t \in [0, 1]$.
2. *the adversary observes the* x_t *and selects a* $b_t \in [0, 1]$
3. *the player receives cost* $f(x_t, b_t) = (1 - \alpha)(x_t - b_t)^2 + \alpha(x_t - s)^2$.

where s, α are constants in $[0, 1]$. The goal of the player is to pick x_t based on the history $(b_0, \ldots, b_{t-1})$ in a way that minimizes her total cost. Generally, different OCO problems can be defined by a set of functions $\mathcal{F}$ that the adversary chooses from and a feasibility set $\mathcal{K}$ from which the player picks her value (see [87] for an introduction to the OCO framework). In our case the feasibility set is $\mathcal{K} = [0, 1]$ and the set of functions is $\mathcal{F}_{s,\alpha} = \{x \mapsto (1 - \alpha)(x - b)^2 + \alpha(x - s)^2 : b \in [0, 1]\}$. As a result, each selection of the constants s, α leads to a different OCO problem.

Definition 3.7. *An algorithm A for the OCO problem with $\mathcal{F}_{s,\alpha}$ and $\mathcal{K} = [0, 1]$ is a sequence of functions $(A_t)_{t=0}^{\infty}$ where $A_t : [0, 1]^t \mapsto [0, 1]$.*

Definition 3.8. *An algorithm A is no-regret for the OCO problem with $\mathcal{F}_{s,\alpha}$ and $\mathcal{K} = [0, 1]$ if and only if for all sequences $(b_t)_{t=0}^{\infty}$ that the adversary may choose, for all $t \geq 1$*

$$\sum_{\tau=0}^{t} f(x_\tau, b_\tau) \leq \min_{x \in [0,1]} \sum_{\tau=0}^{t} f(x, b_\tau) + o(t)$$

where $x_t = A_t(b_0, \ldots, b_{t-1})$

Informally speaking, if the player selects the value x_t according to a *no-regret algorithm* then she does not regret not playing any fixed value no matter what the choices of the adversary are. Theorem 3.2 states that *Follow the Leader* i.e. $x_t = \mathrm{argmin}_{x \in [0,1]} \sum_{\tau=0}^{t-1} f(x, b_\tau)$ is a no-regret algorithm for all the OCO problems with $\mathcal{F}_{s,\alpha}$.

Returning to the dynamics of the game in Definition 3.1, it is reasonable to assume that each agent i selects $x_i(t)$ according to no-regret algorithm A_i

for the OCO problem with $\mathcal{F}_{s_i,\alpha_i}$, since by Definition 3.8,

$$\frac{1}{t}\sum_{\tau=0}^{t} f_i(x_i(\tau), x_{W_i^\tau}(\tau)) \leq \frac{1}{t}\min_{x\in[0,1]}\sum_{\tau=0}^{t} f_i(x, x_{W_i^\tau}(\tau)) + \frac{o(t)}{t}$$

The latter means that the time averaged total disagreement cost that she suffers is close to the time averaged cost by expressing the best fixed opinion and this holds regardless of the opinions of the neighbors that i meets. Meaning that even if the other agents selected their opinions maliciously, her total experienced cost would still be in a sense minimal. Under this perspective update rule (3.2) is a rational choice for selfish agents and as a result FTL dynamics is a *natural* limited information variant of the FJ model.

We now present the key steps for proving Theorem 3.2. We first prove that a similar strategy that also takes into account the value b_t admits no-regret (Lemma 3.3). Obviously, knowing the value b_t before selecting x_t is in direct contrast with the OCO framework, however proving the no-regret property for this algorithm easily extends to establishing the no-regret property of *Follow the Leader*.

Lemma 3.3. *Let* $(b_t)_{t=0}^{\infty}$ *be an arbitrary sequence with* $b_t \in [0,1]$. *Then for all* $t \geq 1$

$$\sum_{\tau=0}^{t} f(y_\tau, b_\tau) \leq \min_{x\in[0,1]}\sum_{\tau=0}^{t} f(x, b_\tau).$$

where $y_t = \text{argmin}_{x\in[0,1]}\sum_{\tau=0}^{t} f(x,b_\tau)$.

Proof. By definition of y_t, $\sum_{\tau=0}^{t} f(y_t, b_\tau) = \min_{x\in[0,1]}\sum_{\tau=0}^{t} f(x, b_\tau)$, so

$$\begin{aligned}\sum_{\tau=0}^{t} f(y_\tau, b_\tau) - \min_{x\in[0,1]}\sum_{\tau=0}^{t} f(x, b_\tau) &= \sum_{\tau=0}^{t} f(y_\tau, b_\tau) - \sum_{\tau=0}^{t} f(y_t, b_\tau)\\ &= \sum_{\tau=0}^{t-1} f(y_\tau, b_\tau) - \sum_{\tau=0}^{t-1} f(y_t, b_\tau)\\ &\leq \sum_{\tau=0}^{t-1} f(y_\tau, b_\tau) - \sum_{\tau=0}^{t-1} f(y_{t-1}, b_\tau)\end{aligned}$$

The last inequality follows by the fact that $y_{t-1} = \text{argmin}_{x\in[0,1]}\sum_{\tau=0}^{t-1} f(x,b_\tau)$ Inductively, we prove that $\sum_{\tau=0}^{t} f(y_\tau, b_\tau) \leq \min_{x\in[0,1]}\sum_{\tau=0}^{t} f(x, b_\tau)$. □

Now we can understand why *Follow the Leader* admits no-regret. Since the cost incurred by the sequence y_t is at most that of the best fixed value, we can compare the cost incurred by x_t with that of y_t. Since the functions in $\mathcal{F}_{s,\alpha}$ are quadratic, the extra term $f(x, b_t)$ that y_t takes into account doesn't change dramatically the minimum of the total sum. Namely, x_t, y_t are relatively close.

Lemma 3.4. *For all $t \geq 0$, $f(x_t, b_t) \leq f(y_t, b_t) + 2\frac{1-\alpha}{t+1} + \frac{(1-\alpha)^2}{(t+1)^2}$.*

Proof. We first prove that for all t,

$$|x_t - y_t| \leq \frac{1-\alpha}{t+1}. \tag{3.10}$$

By definition $x_t = \alpha s + (1-\alpha)\frac{\sum_{\tau=0}^{t-1} b_\tau}{t}$ and $y_t = \alpha s + (1-\alpha)\frac{\sum_{\tau=0}^{t} b_\tau}{t+1}$.

$$\begin{aligned} |x_t - y_t| &= (1-\alpha)\left|\frac{\sum_{\tau=0}^{t-1} b_\tau}{t} - \frac{\sum_{\tau=0}^{t} b_\tau}{t+1}\right| \\ &= (1-\alpha)\left|\frac{\sum_{\tau=0}^{t-1} b_\tau - t b_t}{t(t+1)}\right| \\ &\leq \frac{1-\alpha}{t+1} \end{aligned}$$

The last inequality follows from the fact that $b_\tau \in [0,1]$. We now use inequality (3.10) to bound the difference $f(x_t, b_t) - f(y_t, b_t)$.

$$\begin{aligned} f(x_t, b_t) &= \alpha(x_t - s)^2 + (1-\alpha)(x_t - y_t)^2 \\ &\leq \alpha(y_t - s)^2 + 2\alpha\,|y_t - s|\,|x_t - y_t| + \alpha\,|x_t - y_t|^2 \\ &\quad + (1-\alpha)(y_t - y_t)^2 + 2(1-\alpha)\,|y_t - y_t|\,|x_t - y_t| + (1-\alpha)\,|x_t - y_t|^2 \\ &\leq f(y_t, b_t) + 2\,|x_t - y_t| + |y_t - x_t|^2 \\ &\leq f(y_t, b_t) + 2\frac{1-\alpha}{t+1} + \frac{(1-\alpha)^2}{(t+1)^2} \end{aligned}$$

□

We are now ready to prove Theorem 3.2.

Theorem 3.2. *Consider an arbitrary sequence $(b_t)_{t=0}^{\infty}$ and the function $f : [0,1]^2 \mapsto [0,1]$ with $f(x,b) = (1-\alpha)(x-b)^2 + \alpha(x-s)^2$ for some constants $s, \alpha \in [0,1]$. Then for all $t \geq 0$,*

$$\frac{1}{t}\sum_{\tau=0}^{t} f(x_\tau, b_\tau) \leq \frac{1}{t}\min_{x\in[0,1]}\sum_{\tau=0}^{t} f(x, b_\tau) + O\left(\frac{\log t}{t}\right).$$

where $x_t = \text{argmin}_{x\in[0,1]} \sum_{\tau=0}^{t-1} f(x, b_\tau)$

Proof. Theorem 3.2 easily follows by Lemma 3.3

$$\begin{aligned}
\sum_{\tau=0}^{t} f(x_\tau, b_\tau) &\leq \sum_{\tau=0}^{t} f(y_\tau, b_\tau) + \sum_{\tau=0}^{T} 2\frac{1-\alpha}{\tau+1} + \sum_{\tau=0}^{t} \frac{(1-\alpha)^2}{(\tau+1)^2} \\
&\leq \min_{x\in[0,1]} \sum_{\tau=0}^{t} f(x, y_\tau) + 2(1-\alpha)(\log t + 1) + (1-\alpha)\frac{\pi^2}{6} \\
&\leq \min_{x\in[0,1]} \sum_{\tau=0}^{t} f(x, y_\tau) + O(\log t)
\end{aligned}$$

□

3.5 Lower Bound for Graph Oblivious Dynamics

In this section we prove that any no-regret dynamics cannot converge much faster than FTL dynamics produced by update rule (3.2). This is formally stated in Theorem 3.3 which applies to the more general class of *graph oblivious dynamics*.

Definition 3.9 (no-regret dynamics)**.** *Consider a collection of no-regret algorithms such that for each $(s, \alpha) \in [0,1]^2$ a no-regret algorithm $A_{s,\alpha}$*[1]*for the OCO problem with $\mathcal{F}_{s,\alpha}$ and $\mathcal{K} = [0,1]$, is selected. For a given instance $I = (P, s, \alpha)$ this selection produces the no-regret dynamics $x(t)$ defined as follows:*

- *Initially each agent i selects her opinion $x_i(0) = A_0^{s_i,\alpha_i}(s_i, \alpha_i)$*
- *At round $t \geq 1$, each agent i selects her opinion*

$$x_i(t) = A_t^{s_i,\alpha_i}(x_{W_i^0}(0), \ldots, x_{W_i^{t-1}}(t-1), s_i, \alpha_i)$$

where W_i^t is the neighbors that i meets at round t.

Such a selection of no-regret algorithms can be encoded as a graph oblivious update rule. Specifically, the function $A_t : \{0,1\}^{t+2} \mapsto [0,1]$ is defined as $A_t(b_0, \ldots, b_{t-1}, s, \alpha) = A_{s,\alpha}^t(b_0, \ldots, b_{t-1})$. Thus, Theorem 3.3 applies and establishes the existence of an instance $I = (P, s, \alpha)$ such that the produced $x(t)$ converges at best slowly to x^*. For example if agents use the Online

[1] These s, α are scalars in $[0,1]$ and should not be confused with the internal opinion vector s and the self confidence vector α of an instance $I = (P, s, \alpha)$.

Gradient Descent[2] to update her opinion i.e.

$$x_i(t+1) = x_i(t) - \frac{1}{\sqrt{t}}\left(x_i(t) - (1-\alpha_i)x_{W_i^t}(t) - \alpha_i s_i\right)$$

Then we are ensured that fast convergence cannot be established in the respective no-regret dynamics.

The rest of the section is dedicated to prove Theorem 3.3. In Lemma 5.2 we show that any graph oblivious update rule A can be used as an estimator of the parameter $p \in [0,1]$ of a Bernoulli random variable. Since we prove Theorem 3.3 using a reduction to an estimation problem, we shall first briefly introduce some definitions and notation. For simplicity we will restrict the following definitions of estimators and risk to the case of estimating the parameter p of Bernoulli random variables. Given t independent samples from a Bernoulli random variable $B(p)$, an estimator is an algorithm that takes these samples as input and outputs an answer in $[0,1]$.

Definition 3.10. *An estimator $\theta = (\theta_t)_{t=1}^{\infty}$ is a sequence of functions, $\theta_t : \{0,1\}^t \mapsto [0,1]$.*

Perhaps the first estimator that comes to one's mind is the *sample mean*, that is $\theta_t = \sum_{i=1}^{t} X_i/t$. To measure the efficiency of an estimator we define the *risk*, which corresponds to the expected error of an estimator.

Definition 3.11. *Let P be a Bernoulli distribution with mean p and P^t be the corresponding t-fold product distribution. The risk of an estimator $\theta = (\theta_t)_{t=1}^{\infty}$ is*

$$\mathbf{E}_{(X_1,\ldots,X_t)\sim P^t}\left[|\theta_t(X_1,\ldots,X_t) - p|\right]$$

which we will denote by $\mathbf{E}_p\left[|\theta_t(X_1,\ldots,X_t) - p|\right]$ or $\mathbf{E}_p\left[|\theta_t - p|\right]$ for brevity.

The risk $\mathbf{E}_p\left[|\theta_t - p|\right]$ quantifies the error rate of the estimated value $\hat{p} = \theta_t(Y_1,\ldots,Y_t)$ to the real parameter p as the number of samples t grows. Since p is unknown, any meaningful estimator $\theta = (\theta_t)_{t=1}^{\infty}$ must guarantee that $\lim_{t\to\infty}\mathbf{E}_p\left[|\theta_t - p|\right] = 0$ for all p. For example, *sample mean* has error rate $\mathbf{E}_p\left[|\theta_t - p|\right] \leq \frac{1}{2\sqrt{t}}$.

Lemma 3.5. *Let A a graph oblivious update rule such that for all instances $I = (P, s, \alpha)$,*

$$\lim_{t\to\infty} t^{1+c}\mathbf{E}\left[\|x_A(t) - x^*\|_\infty\right] = 0.$$

[2] Online Gradient Descent is an influential no-regret algorithm proposed by Zinkevic in [142] for the general OCO problem, where the adversary can select any convex function with bounded gradient. The latter directly implies that it also ensures no-regret in our simpler OCO problem with $\mathcal{F}_{s_i,\alpha_i}$ and $\mathcal{K} = [0,1]$.

Then there exists an estimator $\theta_A = (\theta_t^A)_{t=1}^{\infty}$ *such that for all* $p \in [0,1]$, $\lim_{t\to\infty} t^{1+c}\mathbf{E}_p\left[|\theta_t^A - p|\right] = 0$.

Proof. We construct an estimator $\theta_A = (\theta_t^A)_{t=1}^{\infty}$ using the update rule A. Consider the instance I_p described in Figure 3.1. By straightforward computation, we get that the equilibrium point of the graph is $x_c^* = p/3, x_1^* = p/6 + 1/2, x_0^* = p/6$. Now consider the opinion vector $x_A(t)$ produced by the update rule A for the instance I_p. Note that for $t \geq 1$,

- $x_1^A(t) = A_t(x_c(0), \ldots, x_c(t-1), 1, 1/2)$
- $x_0^A(t) = A_t(x_c(0), \ldots, x_c(t-1), 0, 1/2)$
- $x_c^A(t) = A_t(x_{W_c^0}(0), \ldots, x_{W_c^{t-1}}(t-1), 0, 1/2)$

The key observation is that the opinion vector $x_A(t)$ is a deterministic function of the index sequence $W_c^0, \ldots, W_c^{t-1}$ and does not depend on p. Thus, we can construct the estimator θ_A with $\theta_t^A(W_c^0, \ldots, W_c^{t-1}) = 3x_c^A(t)$. For a given instance I_p the choice of neighbor W_c^t is given by the value of the Bernoulli random variable with parameter p ($\mathbf{P}\left[W_c^t = 1\right] = p$). As a result, $\mathbf{E}_p\left[|\theta_t^A - p|\right] = 3\mathbf{E}\left[|x_c^A(t) - p/3|\right] \leq 3\mathbf{E}\left[\|x_A(t) - x^*\|_\infty\right]$. Since for any instance I_p, we have that $\lim_{t\to\infty} t^{1+c}\mathbf{E}\left[\|x_A(t) - x^*\|_\infty\right] = 0$, it follows that $\lim_{t\to\infty} t^{1+c}\mathbf{E}_p\left[|\theta_t^A - p|\right] = 0$ for all $p \in [0,1]$.

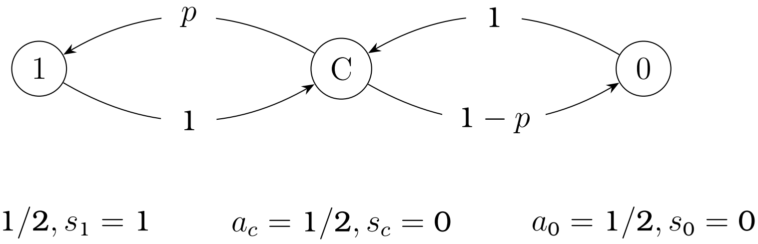


Figure 3.1

□

In order to prove Theorem 3.3 we just need to prove the following claim.

Claim 3.1. *For any estimator* $\theta = (\theta_t)_{t=1}^{\infty}$ *there exists a* $p \in [0,1]$ *such that* $\lim_{t\to\infty} t^{1+c}\mathbf{E}_p\left[|\theta_t - p|\right] > 0$.

The above claim states that for any estimator $\theta = (\theta_t)_{t=1}^{\infty}$, we can inspect the functions $\theta_t : \{0,1\}^t \mapsto [0,1]$ and then choose a $p \in [0,1]$ such that the function $\mathbf{E}_p\left[|\theta_t - p|\right] = \Omega(1/t^{1+c})$. As a result, we have reduced the construction of a lower bound concerning the round complexity of a dynamical process to a lower bound concerning the sample complexity of estimating the parameter p of a Bernoulli distribution. The claim follows by Lemma 3.6, which we present at the end of the section.

At this point we should mention that it is known that $\Omega(1/\varepsilon^2)$ samples are needed to estimate the parameter p of a Bernoulli random variable within additive error ε. Another well-known result is that taking the average of the samples is the *best* way to estimate the mean of a Bernoulli random variable. These results would indicate that the best possible rate of convergence for an *graph oblivious dynamics* would be $O(1/\sqrt{t})$. However, there is some fine print in these results which does not allow us to use them. In order to explain the various limitations of these methods and results we will briefly discuss some of them. We remark that this discussion is not needed to understand the proof of Lemma 3.6.

The oldest sample complexity lower bound for estimation problems is the well-known Cramer-Rao inequality. Let the function $\theta_t : \{0,1\}^t \mapsto [0,1]$ such that $\mathbf{E}_p[\theta_t] = p$ for all $p \in [0,1]$, then

$$\mathbf{E}_p\left[(\theta_t - p)^2\right] \geq \frac{p(1-p)}{t}. \tag{3.11}$$

Since $\mathbf{E}_p[|\theta_t - p|]$ can be lower bounded by $\mathbf{E}_p[(\theta_t - p)^2]$ we can apply the Cramer-Rao inequality and prove our claim in the case of *unbiased* estimators, $\mathbf{E}_p[\theta_t] = p$ for all t. Obviously, we need to prove it for any estimator θ, however this is a first indication that our claim holds.

Sample complexity lower bounds without assumptions about the estimator are usually given as lower bounds for the *minimax risk*, which was defined [3] by Wald in [138] as

$$\min_{\theta_t} \max_{p \in [0,1]} \mathbf{E}_p[|\theta_t - p|].$$

Minimax risk captures the idea that after we pick the best possible algorithm, an adversary inspects it and picks the worst possible $p \in [0,1]$ to generate the samples that our algorithm will get as input. The methods of Le'Cam, Fano, and Assouad are well-known information-theoretic methods to establish lower bounds for the minimax risk. For more on these methods see [141, 136]. As we stated before, it is well known that the minimax risk for the case of estimating the mean of a Bernoulli is lower bounded by $\Omega(1/\sqrt{t})$ and this lower bound can be established by Le Cam's method. In order to show why such results do no work for our purposes we shall sketch how one would apply Le Cam's method to get this lower bound. To apply Le Cam's method, one typically chooses two Bernoulli distributions whose means are far but their

[3] Although the minimax risk is defined for any estimation problem and loss function, for simplicity, we write the minimax risk for estimating the mean of a Bernoulli random variable.

total variation distance is small. Le Cam showed that when two distributions are close in total variation then given a sequence of samples $X_1, \ldots, X_t$ it is hard to tell whether these samples were produced by P_1 or P_2. The hardness of this *testing* problem implies the hardness of *estimating* the parameters of a family of distributions. For our problem the two distributions would be $B(1/2 - 1/\sqrt{t})$ and $B(1/2 + 1/\sqrt{t})$. It is not hard to see that their total variation distance is at most $O(1/t)$, which implies a lower bound $\Omega(1/\sqrt{t})$ for the minimax risk. The problem here is that the parameters of the two distributions depend on the number of samples t. The more samples the algorithm gets to see, the closer the adversary takes the 2 distributions to be. For our problem we would like to *fix* an instance and then argue about the rate of convergence of any algorithm on this instance. Namely, having an instance that depends on t does not work for us.

Trying to get a lower bound without assumptions about the estimators while respecting our need for a fixed (independent of t) p we prove Lemma 3.6. In fact, we show something stronger: for *almost all* $p \in [0, 1]$, any estimator θ cannot achieve rate $o(1/t^{1+c})$. More precisely, suppose we select p uniformly at random in $[0, 1]$ and run the estimator θ with samples from the distribution $B(p)$, then with probability 1 the error rate $\mathbf{E}_p\left[|\theta_t - p|\right] = \Omega(1/t^{1+c})$.

Lemma 3.6. *Let $\theta = (\theta_t)_{t=1}^{\infty}$ be a Bernoulli estimator with error rate $\mathbf{E}_p\left[|\theta_t - p|\right]$. For any $c > 0$, if we select p uniformly at random in $[0, 1]$ then*

$$\lim_{t\to\infty} t^{1+c}\mathbf{E}_p\left[|\theta_t - p|\right] > 0$$

with probability 1.

Proof. Since θ_t is a function from $\{0, 1\}^t$ to $[0, 1]$, θ_t can have at most 2^t different values. Without loss of generality, we assume that θ_t takes the same value $\theta_t(x)$ for all $x \in \{0, 1\}^t$ with the same number of 1's. For example, $\theta_3(\{1, 0, 0\}) = \theta_3(\{0, 1, 0\}) = \theta_3(\{0, 0, 1\})$. This is due to the fact that for any $p \in [0, 1]$,

$$\sum_{0\le i\le t} \sum_{\|x\|_1 = i} |\theta_t(x) - p|\, p^i(1-p)^{t-i} \ge \sum_{0\le i\le t} \binom{t}{i} \left| \frac{\sum_{\|x\|_1=i} \theta_t(x)}{\binom{t}{i}} - p \right| p^i(1-p)^{t-i}.$$

For any estimator θ with error rate $\mathbf{E}_p\left[|\theta_t - p|\right]$ there exists another estimator θ' that satisfies the above property and $\mathbf{E}_p\left[|\theta'_t - p|\right] \le \mathbf{E}_p\left[|\theta_t - p|\right]$ for all $p \in [0, 1]$. Thus, we can assume that θ_t takes at most $t + 1$ different values. Let A denote the set of p for which the estimator has error rate $o(1/t^{1+c})$, that is

$$A = \{p \in [0, 1] : \lim_{t\to\infty} t^{1+c}\mathbf{E}_p\left[|\theta_t - p|\right] = 0\}.$$

We show that if we select p uniformly at random in $[0,1]$ then $\mathbf{P}[p \in A] = 0$. We also define the set

$$A_k = \{p \in [0,1] : \text{for all } t \geq k,\ t^{1+c}\mathbf{E}_p[|\theta_t - p|] \leq 1\}.$$

Observe that if $p \in A$ then there exists t_p such that $p \in A_{t_p}$, meaning that $A \subseteq \bigcup_{k=1}^{\infty} A_k$. As a result,

$$\mathbf{P}[p \in A] \leq \mathbf{P}\left[p \in \bigcup_{k=1}^{\infty} A_k\right] \leq \sum_{k=1}^{\infty} \mathbf{P}[p \in A_k].$$

To complete the proof we show that $\mathbf{P}[p \in A_k] = 0$ for all k. Notice that $p \in A_k$ implies that for $t \geq k$, the estimator θ must always have a value $\theta_t(i)$ close to p. Using this intuition we define the set

$$B_k = \{p \in [0,1] : \text{for all } t \geq k,\ t^{1+c} \min_{0 \leq i \leq t} |\theta_t(i) - p| \leq 1\}.$$

We now show that $A_k \subseteq B_k$. Since $p \in A_k$ we have that for all $t \geq k$

$$\begin{aligned} t^{1+c} \min_{0 \leq i \leq t} |\theta_t(i) - p| \sum_{i=0}^{t} \binom{t}{i} p^i (1-p)^{t-i} &\leq t^{1+c} \sum_{i=0}^{t} \binom{t}{i} |\theta_t(i) - p|\, p^i (1-p)^{t-i} \\ &= t^{1+c}\mathbf{E}_p[|\theta_t - p|] \\ &= 1 \end{aligned}$$

Thus, $\mathbf{P}[p \in A_k] \leq \mathbf{P}[p \in B_k]$. We write the set B_k as

$$B_k = \bigcap_{t=k}^{\infty} \{p \in [0,1] : \min_{0 \leq i \leq t} |\theta_t(i) - p| \leq 1/t^{1+c}\}.$$

As a result,

$$\mathbf{P}[p \in B_k] \leq \mathbf{P}\left[\min_{0 \leq i \leq t} |\theta_t(i) - p| \leq 1/t^{1+c}\right], \text{for all } t \geq k.$$

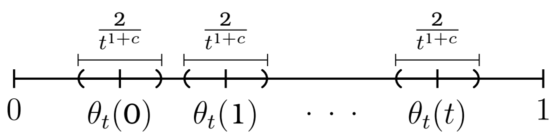


Figure 3.2: Estimator output at time t

Each value $\theta_t(i)$ «covers» length $1/t^{1+c}$ from its left and right, as shown in Figure 3.2, and since there are at most $t+1$ such values, by the union bound we get

$$\mathbf{P}[p \in B_k] \leq 2(t+1)/t^{1+c} \text{ for all } t \geq k$$

We conclude that $\mathbf{P}[p \in B_k] = 0$. □

Remark 3.1. *The only point that we use that the update rules are* graph oblivious *is in Lemma 5.2. It is not difficult to see that the reduction still holds if the update rules also depend on the indices of the neighbors that an agent meets. As a result, Theorem 3.3 still applies.*

3.6 Limited Information Dynamics with Fast Convergence

We already discussed that the reason that graph oblivious dynamics suffer slow convergence is that the update rule depends only on the observed opinions. Based on works for asynchronous distributed minimization algorithms [13, 42], we provide an update rule showing that information about the graph G combined with agents that do not act selfishly can restore the fast convergence rate (update rule (3.12)). Our update rule depends not only on the expressed opinions of the neighbors that an agent i meets, but also on the i-th row of matrix P. We have already mentioned that while this update rule guarantees fast convergence it does not guarantee no-regret to the agents, see Example 3.1. Agents that select their opinions according to this rule may experience regret if some other agents play adversarially.

In update rule (3.12), each agent stores the *most recent* opinions of the random neighbors that she meets in an array and then updates her opinion according to their weighted sum (each agent knows row i of P). For a given instance $I = (P, s, \alpha)$ we call the produced dynamics *Row Dependent dynamics*.

Row Dependent dynamics

Initially $x_i(0) = s_i$ for all agent i.
Each agent i keeps an array M_i of length $|N_i|$, randomly initialized.
At round $t \geq 0$ each agent i:
Meets neighbor with index W_i^t where $\mathbf{P}\left[W_i^t = j\right] = p_{ij}$
Suffers disagreement cost

$$(1-\alpha_i)(x_i(t) - x_{W_i^t}(t))^2 + a_i(x_i(t) - s_i)^2$$

and learns the opinion $x_{W_i^t}(t)$.
Updates her array M_i and her opinion as follows:

$$M_i[W_i^t] \leftarrow x_{W_i^t}(t)$$

$$x_i(t+1) = (1-\alpha_i)\sum_{j \neq i} p_{ij} M_i[j] + \alpha_i s_i \tag{3.12}$$

The problem with this approach is that the opinions of the neighbors that she keeps in her array are *outdated*, i.e. the opinion of a neighbor of agent i has changed since their last meeting. The good news are that as long as this outdatedness is bounded we can still achieve fast convergence to the equilibrium. By bounded outdatedness we mean that there exists a number of rounds B such that all agents have met all their neighbors at least once from $t - B$ to t. The latter is formally stated in Lemma 3.7.

Remark 3.2. *Update rule (3.12), apart from the opinions and the indices of the neighbors that an agent meets, also depends on the the exact values of the weights p_{ij} and that is why* Row Dependent dynamics *converge fast. We mention that the lower bound of Section 3.5 still holds even if the agents also use the indices of the neighbors that they meet to update their opinion, since Lemma 5.2 can be easily modified to cover this case. The latter implies that any update rule that ensures fast convergence would require from each agent i to be* aware *of the i-th row of matrix P.*

Lemma 3.7. *Let $\rho = \min_i a_i$, and $\pi_{ij}(t)$ be the most recent round before round t, that agent i met her neighbor j. If for all $t \geq B$, $t - B \leq \pi_{ij}(t)$ then, for all $t \geq kB$,*

$$\|x(t) - x^*\|_\infty \leq (1 - \rho)^k$$

In Row Dependent dynamics there does not exist a fixed length window B that satisfies the requirements of Lemma 3.7. However we can select a length value such that the requirements hold with high probability. To do this observe that agent i simply needs to wait to meet the neighbor j with the smallest weight p_{ij}. Therefore, after $\log(1/\delta)/\min_j p_{ij}$ rounds we have that with probability at least $1 - \delta$ agent i met all her neighbors at least once. Since we want this to be true for all agents, we shall roughly take $B = 1/\min_{p_{ij}>0} p_{ij}$.

In the rest of the section we give the detailed argument that leads to Theorem 3.6, showing that the convergence rate of update rule (3.12) is fast.

Theorem 3.6. *Let $I = (P, s, \alpha)$ be an instance of the opinion formation game of Definition 3.1 with equilibrium $x^* \in [0,1]^n$ and let $\rho = \min_{i \in V} a_i$. The opinion vector $x(t) \in [0,1]^n$ produced by update rule (3.12) after t rounds satisfies*

$$\mathbf{E}\left[\|x(t) - x^*\|_\infty\right] \leq 2e^{-\rho \min_{ij} p_{ij} \sqrt{t}/(4\ln(nt))}.$$

We are now going to state and prove a series of lemmas that culminate in the proof of Theorem 3.6.

Proof of Lemma 3.7. To prove our claim we use induction on k. For the induction base $k = 1$,

$$\begin{aligned} |x_i(t) - x_i^*| &= |(1-\alpha_i)\sum_{j\neq i} p_{ij}(x_j(\pi_{ij}(t)) - x_j^*)| \\ &\leq (1-\alpha_i)\sum_{j\neq i} p_{ij}|x_j(\pi_{ij}(t)) - x_j^*| \\ &\leq (1-\rho) \end{aligned}$$

Assume that for all $t \geq (k-1)B$ we have that $\|x(t) - x^*\|_\infty \leq (1-\rho)^{k-1}$. For $k \geq 2$, we again have that

$$|x_i(t) - x_i^*| \leq (1-\rho)\sum_{j\neq i} p_{ij}|x_j(\pi_{ij}(t)) - x_j^*|$$

Since $t - B \leq \pi_{ij}(t)$ and $t \geq kB$ we obtain that $\pi_{ij}(t) \geq (k-1)B$. As a result, the inductive hypothesis applies, $|x_j(\pi_{ij}(t)) - x_j^*| \leq (1-\rho)^{k-1}$ and $|x_i(t) - x_i^*| \leq (1-\rho)^k$. □

We now turn our attention to the problem of calculating the size of window B, such that with high probability all agents have outdatedness at most B. We first state a useful fact concerning the coupons collector problem.

Lemma 3.8. *Suppose that the collector picks coupons with different probabilities, where n is the number of distinct coupons. Let w be the minimum of these probabilities. If he selects $\ln n/w + c/w$ coupons, then:*

$$\mathbf{P}\left[\textit{collector hasn't seen all coupons}\right] \leq \frac{1}{\mathrm{e}^c}$$

Lemma 3.9. *Let $\pi_{ij}(t)$ be the most recent round before round t that agent i met agent j and $B = 2\ln(\frac{nt}{\delta})/\min_{ij} p_{ij}$. Then with probability at least $1-\delta$, for all $\tau \geq B$ and for all $i, j \neq i$*

$$\tau - B \leq \pi_{ij}(\tau) \leq \tau - 1.$$

Proof. For simplicity we denote $w = \min_{ij} p_{ij}$. Consider an agent i at round $\tau \geq B$ where $B = 2\ln(\frac{nt}{\delta})/w$ and assume that there exists an agent $j \in N_i$ such that $\pi_{ij}(\tau) < \tau - B$. Agent i can be viewed as a coupon collector that has buyed B coupons but has not found the coupon corresponding to agent j. Since $N_i < n$ and $\min_{j\neq i} p_{ij} \geq w$ by Lemma 3.8 we have that

$$\mathbf{P}\left[\text{there exists } j \in N_i \text{ s.t. } \pi_{ij}(\tau) < \tau - B\right] \leq \frac{\delta}{nt}$$

The proof follows by a union bound for all agent i and all round $B \leq \tau \leq t$. □

By direct application of Lemma 3.7 and Lemma 3.9, we obtain the following corollary that will be useful in proving Theorem 3.6.

Corollary 3.2. *Let $x(t)$ the opinion vector produced by update rule (3.12) for the instance $I = (P, s, \alpha)$, then with probability at least $1 - \delta$*

$$\|x(t) - x^*\|_\infty \leq \mathrm{e}\left(-\frac{\rho t \min_{ij} p_{ij}}{2\ln(\frac{nt}{\delta})}\right)$$

where $\rho = \min_{i \in V} \alpha_i$.

Proof. Let $B = 2\ln(\frac{nt}{\delta})/\min_{ij} p_{ij}$. By Lemma 3.9 we have that with probability at least $1 - \delta$, for all $i, j \in N_i$ and for all $\tau \geq B$,

$$\tau - B \leq \pi_{ij}(\tau)$$

As a result, with probability at least $1 - \delta$ the requirements of Lemma 3.7 are satisfied, meaning that

$$\|x(t) - x^*\|_\infty \leq (1 - \rho)^{\frac{t}{B}} \leq \mathrm{e}\left(-\frac{\rho t \min_{ij} p_{ij}}{2\ln(\frac{nt}{\delta})}\right)$$

□

We can now prove Theorem 3.6 using the previous results.

Theorem 3.6. *Let $I = (P, s, \alpha)$ be an instance of the opinion formation game of Definition 3.1 with equilibrium $x^* \in [0, 1]^n$ and let $\rho = \min_{i \in V} a_i$. The opinion vector $x(t) \in [0, 1]^n$ produced by update rule (3.12) after t rounds satisfies*

$$\mathbf{E}\left[\|x(t) - x^*\|_\infty\right] \leq 2e^{-\rho \min_{ij} p_{ij} \sqrt{t}/(4\ln(nt))}.$$

Proof. Let $u(t) = \|x(t) - x^*\|_\infty$ and $w = \min_{ij} p_{ij}$. From Corollary 3.2 we obtain:

$$\mathbf{P}\left[u(t) > \mathrm{e}\left(-\frac{\rho w t}{2\ln(\frac{nt}{\delta})}\right)\right] \leq \delta$$

for every probability $\delta \in [0, 1]$. Also, since all the parameters of the problem lie in $[0, 1]$, we have

$$\mathbf{E}\left[u(t) | u(t) > r\right] \leq 1$$

Now, by the conditional expectations identity, we get:

$$\begin{aligned}\mathbf{E}\left[u(t)\right] &= \mathbf{E}\left[u(t)|u(t) > r\right]\mathbf{P}\left[u(t) > r\right] + \mathbf{E}\left[u(t)|u(t) \leq r\right]\mathbf{P}\left[u(t) \leq r\right] \\ &\leq \delta + r\end{aligned}$$

where $r = \mathrm{e}\left(-\frac{\rho w t}{2\ln(\frac{nt}{\delta})}\right)$. If we set $\delta = \mathrm{e}\left(-\frac{\rho w\sqrt{t}}{2\ln nt}\right)$, then:

$$\mathbf{E}\left[u(t)\right] \leq \mathrm{e}\left(-\frac{\rho w\sqrt{t}}{2\ln nt}\right) + \mathrm{e}\left(-\frac{\rho w t}{2\ln(\frac{nt}{\delta})}\right)$$

We now evaluate r for our choice of probability δ:

$$\begin{aligned}
r &= \mathrm{e}\left(-\frac{\rho w t}{2\ln\left(\frac{nt}{p}\right)}\right) \\
&= \mathrm{e}\left(-\frac{\rho w t}{2\ln\left(\frac{nt}{\mathrm{e}\left(-\frac{\rho w\sqrt{t}}{2\ln nt}\right)}\right)}\right) \\
&= \mathrm{e}\left(-\frac{\rho w t}{2\ln nt + 2\frac{\rho w\sqrt{t}}{2\ln nt}}\right) \\
&\leq \mathrm{e}\left(-\frac{\rho w t}{4\ln(nt)\sqrt{t}}\right) \\
&= \mathrm{e}\left(-\frac{\rho w\sqrt{t}}{4\ln(nt)}\right)
\end{aligned}$$

Using the previous calculation, we obtain:

$$\begin{aligned}
\mathbf{E}\left[u(t)\right] &\leq \mathrm{e}\left(-\frac{\rho w\sqrt{t}}{2\ln(nt)}\right) + \mathrm{e}\left(-\frac{\rho w\sqrt{t}}{4\ln(nt)}\right) \\
&\leq 2\mathrm{e}\left(-\frac{\rho w\sqrt{t}}{4\ln(nt)}\right) \\
&= 2\mathrm{e}\left(-\rho\min_{ij} p_{ij}\frac{\sqrt{t}}{4\ln(nt)}\right)
\end{aligned}$$

□

Example 3.1. *The purpose of this example is to illustrate that the update rule (3.12) does not ensure the no-regret property. If some agents for various reasons exhibit irrational or adversarial behavior, agents that adopt update rule (3.12) may experience regret. That is the reason that* Row Dependent dynamics *converge exponetially faster that any* no-regret dynamics *incluing the* FTL dynamics*.*

Consider the instance of the game of Definition 3.1 consisting of two agents. Agent 1 *adopts update rule (3.12) and has* $s_1 = 0, \alpha_1 = 1/2, p_{12} = 1$ *and agent* 2 *plays adversarially. Thus,* s_2, α_2, p_{21} *don't need to be specified. By update rule (3.12),* $x_1(t) = x_2(t-1)/2$ *and thus total disagreement cost that agent* 1 *experiences until round* t *is*

$$\sum_{\tau=0}^{t} \frac{1}{2} x_1(t)^2 + \frac{1}{2}(x_1(t) - x_2(t))^2 = \sum_{\tau=0}^{t} \frac{1}{8} x_2(t-1)^2 + \frac{1}{2}(\frac{1}{2} x_2(t-1) - x_2(t))^2.$$

Since agent 2 *plays adversarially, she selects* $x_2(t) = 0$ *if* t *is even and* 1 *otherwise. As a result, the total cost that agent* 1 *experiences is* $\sum_{\tau=0}^{t} \frac{1}{2} x_1(t)^2 + \frac{1}{2}(x_1(t) - x_2(t))^2 \simeq 3t/8$. *Now agent* 1 *regrets for no adopting the fixed opinion* $1/3$ *during the whole game play. Selecting* $x_1(t) = 1/3$ *for all* t*, would incur him total disagreement cost* $\sum_{\tau=0}^{t} \frac{1}{2}(1/3)^2 + \frac{1}{2}(1/3 - x_2(t))^2 \simeq 7t/36$ *which is less than* $3t/8$.

Chapter 4

Opinion Formation with Aggregation and Negative Influence

In this chapter, we present our results on the average-oriented opinion formation games. This kind of games consist a generalization of the opinion formation games [17] and were introduced in our work [64] in order to capture the fact that opinions are oftently affected by global social trends and phenomena. A more comprehensive introduction on this kind of games can be found in Section 2.5.2.

4.1 Average-Oriented Opinion Formation Games

Average-Oriented opinion formation games consist of n selfish agents in which the strategy of each selfish agent i, is the opinion x_i that she publicly expresses. For a given opinion vector $x = (x_i, x_{-i})$, each agent i experiences disagreement cost

$$C_i(x) = \sum_{j \neq i} w_{ij}(x_j - x_i)^2 + w_i(x_i - s_i)^2 + \alpha_i(\mathsf{avg}(x) - s_i)^2 \tag{4.1}$$

where

- $s_i \in [0, 1]$ denotes the internal opinion of agent i.
- $\mathsf{avg}(x)$ is the average public opinion, $\mathsf{avg}(x) = \sum_{j=1}^{n} x_j / n$.
- $w_i > 0$ is the self-confidence of agent i towards her internal opinion s_i.

- $w_{ij} \geq 0$ is the influence that agent j poses on i.
- $\alpha_i \geq 0$ measures i's sensitivity towards the average opinion.

As already discussed, the opinion formation game of Equation 2.1 ([17]) is a special case of the average-oriented opinion formation games of Equation 4.1 where $\alpha_i = 0$ for all agents. While the simultaneous best response dynamics of the average-opinion formation games is a generalization of the FJ model (consider Equation 4.2 with $\alpha_i = 0$). For a given opinion vector $x(t) = (x_i(t), x_{-i}(t))$ at round t, agent's i best response is given by Equation 4.2.

$$x_i(t+1) = \frac{\sum_{j\neq i}\left(w_{ij} - \frac{\alpha_i}{n^2}\right)x_j(t) + \left(w_i + \frac{\alpha_i}{n}\right)s_i}{w_i + \frac{\alpha_i}{n^2} + \sum_{j\neq i} w_{ij}} \quad . \tag{4.2}$$

In with chapter we investigate how the aggregation term $\alpha_i(\mathsf{avg}(x) - s_i)^2$ affects the convergence properties of the FJ model and the efficiency of equilibrium in terms of total disagreement cost.

4.2 Our Results

In Section 2.3 we saw that the FJ model converges to the Nash Equilibrium of the orginal opinion formation game [17]. The basic challenge in generalizing these convergece resuls in the average-oriented opinion formation games stems from the fact that i's influence from some opinions x_j can be negative (see Equation 4.2). Negative influence is introduced due to the competition of the agents for dragging the average public opinion close to their intrinsic beliefs.

Despite negative influence, we show that if agents admit a certain level of *self-confidence* w_i, simultaneous best-response dynamics in average-oriented opinion formation games converges fast to the Nash equilibrium of the game. We should highlight that assuming positive self-confidence is necessary for convergence [79, 81] and that the convergence time decreases as the ratio of w_i to α_i and to $\sum_{j\neq i} w_{ij}/(n-1)$ increases. For clarity, we make the reasonable assumption that $w_i \geq \alpha_i$ and $w_i \geq \sum_{j\neq i} w_{ij}/(n-1)$. Namely, we assume that the self-confidence level of each agent is no less than her influence from the average public opinion and no less than her average influence from other agents (this is also consistent with the confidence level assumed in [81]). Under this condition, we show that simultaneous best-response dynamics in average-oriented opinion formation games converges to the unique Nash Equilibrium within distance $\varepsilon > 0$ in $O(n^2 \log(n/\varepsilon))$ rounds (Lemma 4.1).

Simultaneous best response dynamics assumes that all agents have access to the average public opinion in each round in order to compute their best response. Since the average public opinion is global information and thus expensive to obtain in large networks, we consider average-oriented opinion dynamics with outdated information. Here the average public opinion is announced to all the agents simultaneously every few rounds (e.g. a polling agency publishes this information in a web page now and then). We prove (Theorem 4.1) that opinion dynamics with outdated information about the average public opinion converges to the Nash equilibrium of the game within distance $\varepsilon > 0$ after $O(n^2 \log(n/\varepsilon))$ «announcements» of the average public opinion. Both these results are proven for a more general setting with negative influence between the agents and with partially outdated information about the agents' public opinions. We essentially prove that negative influence and outdated information do not introduce undesirable oscillating phenomena to opinion dynamics. Our proofs make use of matrix norm properties, which allow us to deal with negative influence between the agents and with the difficulties introduced by outdated information.

In Section 4.5, we bound the PoA of average-oriented opinion formation games. We restrict our attention to the most interesting case of symmetric games, where $w_{ij} = w_{ji}$, all agents have the same self-confidence w and the same influence α from the average. For nonsymmetric games the PoA is $\Omega(n)$, even without aggregation if $\alpha = 0$ [17]. We show (Theorem 4.2) that the PoA is at most $9/8 + O(\alpha/(wn^2))$. In general, this bound cannot be improved since for $\alpha = 0$, $9/8$ is a tight bound for the PoA [17]. Our proof builds on the elegant local smoothness approach of [15]. However, local smoothness cannot be directly applied to symmetric average-oriented games, because the function $(\mathsf{avg}(x) - s_i)^2$ is not locally smooth. To overcome this difficulty, we carefully combine local smoothness with the fact that the average public opinion at equilibrium is equal to the average intrinsic belief, a consequence of symmetry (Proposition 4.2).

A frequent assumption in the literature on continuous opinion formation is that agent beliefs and opinions take values in a finite interval of non-negative real numbers. Then, by scaling, one can assume that beliefs and opinions lie in $[0, 1]$. Thus, we always assume that agent beliefs $s_i \in [0, 1]$. On the other hand, an important side-effect of negative influence is that the best-response (and in Nash Equilibrium) opinions may become polarized and be pushed towards opposite directions, far away from $[0, 1]$. We believe that such opinion polarization is natural and should be allowed when negative influence is considered. Therefore, in Sections 4.4 and 4.5, we assume that the public opinions take arbitrary real values. Then, in Section 4.6, we consider *restricted* average-oriented games, where the strategy space of the agents is resctricted to $[0, 1]$, and study how convergence properties and the price of anarchy are affected.

Existence and uniqueness of equilibrium for restricted opinion formation games follow from [126]. We prove (Lemma 4.3 and Theorem 4.3) that the convergence rate of opinion dynamics with negative influence and with outdated information is not affected by restriction of public opinions to $[0, 1]$. The analysis of the convergence rate is similar to that for (unrestricted) opinion formation games. The only difference is a simple case analysis, in the final part of the proofs of Lemma 4.3 and Theorem 4.3, which establishes that the distance of the restricted opinion vector to equilibrium decreases at least as fast as the corresponding distance in the unrestricted case.

For the PoA of restricted symmetric games, we consider the special case where $w = \alpha = 1$ and show that the $\mathsf{PoA} \leq 3 + 2\sqrt{2} + O(\frac{1}{n})$ (Theorem 4.4). The main technical challenge in the PoA analysis of restricted games is that the local smoothness argument of Theorem 4.2 does not apply, because the function $(\mathsf{avg}(x) - s_i)^2$ is not locally smooth and the average public opinion at equilibrium may be far from the average intrinsic belief. Hence, in the proof of Theorem 4.4, we need to advance substantially beyond the local smoothness argument of Theorem 4.2. More specifically, we first show that if all agents only value the distance of their opinion to the average and to their belief ($w_{ij} = 0$) the PoA is at most $1 + 1/n^2$ (Proposition 4.7). Then we combine the PoA of this simpler game with the local smoothness inequality of [15] and bound the PoA of restricted symmetric games.

4.3 Definitions and Preliminaries

For an $n \times n$ matrix A, $\|A\| = \max_{i \in N} \sum_{j=1}^{n} |a_{ij}|$ is the infinity norm of A and capital N denotes the set $\{1, n\}$. Similarly, for an n-dimensional vector x, $\|x\| = \max_{i \in N} |x_i|$ is the infinity norm of x. We use the standard properties of matrix norms without explicitly referring to them. Specifically, we use that

- for any matrices A and B, $\|AB\| \leq \|A\| \, \|B\|$ and $\|A+B\| \leq \|A\| + \|B\|$
- for any matrix A and any $\lambda \in \mathbf{R}$, $\|\lambda A\| \leq |\lambda| \, \|A\|$
- for any matrix A and any integer ℓ, $\|A^\ell\| \leq \|A\|^\ell$

Moreover, we use that for any $n \times n$ real matrix A with $\|A\| < 1$,

$$\sum_{\ell=0}^{\infty} A^\ell = (\mathbb{I} - A)^{-1}$$

4.3.1 Average-Oriented Opinion Formation

Without loss of generality, we assume that the vector of agent beliefs s lies in $[0,1]^n$. As for the public opinions x, we initially assume values in $\mathbf{R}$ and then, in Section 4.6, explain what changes if we restrict them to $[0,1]$.

Definition 4.1. *An average-oriented opinion formation game $\mathcal{G}$ is* symmetric *if*

1. $w_{ij} = w_{ji}$ *for all* $i \neq j$.
2. $w_i = w$ *for all agents* i.
3. $\alpha_i = \alpha$ *for all agents* i

If Definition 4.1 is not satisfied by $\mathcal{G}$, we call it *nonsymmetric*. Our convergence results hold for nonsymmetric games, while the **PoA** bounds hold only for symmetric ones.

An opinion vector x^* is a *Nash equilibrium* the agents cannot improve on their individual cost by unilaterally changing their opinions.

Definition 4.2. *An opinion vector $x^* = (x^*_i, x^*_{-i}) \in [0,1]^n$ is a Nash equilibrium of an opinion formation game. $\mathcal{G}$ if for each agent i*

$$C_i(x^*) \leq C_i(x_i, x^*_{-i}) \text{ for all } x_i \in \mathbf{R}$$

We highlight that not all opinion formation games $\mathcal{G}$ admit Nash equilibrium (see Example 2.3). However if an average-oriented opinion formation game $\mathcal{G}$ admits Nash Equilibrium x^*, then this must be unique since it must be the solution of the linear system of Equation 4.2.

Definition 4.3. *The* Price of Anarchy *of an opinion formation game $\mathcal{G}$ with Nash Equilibrium x^* is,*

$$\mathrm{PoA}(\mathcal{G}) = \frac{\sum_{i\in N} C_i(x^*)}{\sum_{i\in N} C_i o^*)}$$

where o^* is the minimizer of $\sum_{i\in N} C_i(x)$.

To study the convergence properties of simultaneous best-response dynamics, it is convenient to write (4.2) in matrix form. Let $S_i = w_i + \frac{\alpha_i}{n^2} + \sum_{j\neq i} w_{ij}$. We define two $n \times n$ matrices A and B. Matrix A has $a_{ii} = 0$, for all $i \in N$, and $a_{ij} = (w_{ij} - \frac{\alpha_i}{n^2})/S_i$, for all $j \neq i$. Matrix B is diagonal and has $b_{ii} = (w_i + \frac{\alpha_i}{n})/S_i$, for all $i \in N$, and $b_{ij} = 0$, for all $j \neq i$. Note

$\frac{d^2C_i(x)}{dx_i^2} = 2S_i > 0$ and thus function $C_i(x)$ is strictly convex in x_i, even if some entries of A are negative.

The simultaneous best-response dynamics of an average-oriented game $\mathcal{G}$ starts with $x(0) = s$ and proceeds in rounds. In each round $t \geq 1$, the public opinion vector $x(t)$ is:

$$x(t) = Ax(t-1) + Bs\,. \tag{4.3}$$

We say that simultaneous best response dynamics *converges* to Nash equilibrium x^* if for all $\varepsilon > 0$, there is a $t^*(\varepsilon)$, such that for all $t \geq t^*(\varepsilon)$,

$$\|x(t) - x^*\| \leq \varepsilon$$

Iterating (4.3) over t, we obtain that for all rounds $t \geq 1$,

$$x(t) = Ax(t-1)+Bs = A(Ax(t-2)+Bs)+Bs = \cdots = A^t s+\sum_{\ell=0}^{t-1} A^\ell Bs \tag{4.4}$$

Since convergence of the simultaneous best response dynamics implies the existence of Nash equilibrium x^* which may do not exist (Example 2.3), some necessary assumption must be made. Assuming that agent self-confidence levels w_i are positive is necessary for convergence of the simultaneous best response dynamics (consider an opinion formation game where $w_i = \alpha_i = 0$ for all agents i and the matrix A corresponds to the adjacency matrix of a bipartite network). Similarly to [81] and for clarity we make the following assumption.

Assumption 1. *For each agent i:*

1. *the self-confidence level w_i is at least as large as her average influence from other agents*
$$w_i \geq \frac{\sum_{j \neq i} w_{ij}}{n-1}$$
2. *the self-confidence level of any agent is no less than her influence from the average public opinion*
$$w_i \geq \alpha_i$$

Assumption 1 immediately implies that for any agent i,

$$\alpha_i \leq S_i \leq (n + \frac{1}{n^2})nw_i$$

Using this inequality on S_i, we obtain the following inequality, which is crucial for the convergence rate of simultaneous best response dynamics:

$$
\begin{aligned}
\|A\| &\leq \frac{S_i - w_i + \frac{\alpha_i(n-2)}{n^2}}{S_i} \\
&\leq \frac{S_i - \frac{S_i n^2}{n^3+1} + \frac{S_i(n-2)}{n^2}}{S_i} \\
&\leq 1 - \frac{2}{n^2} + \frac{1}{n^3} \\
&\leq 1 - \frac{1}{n^2} \qquad (4.5)
\end{aligned}
$$

We use (4.5) in Corollary 4.1 and Corollary 4.2 and show that the best response dynamics converges to equilibrium within distance $\varepsilon > 0$ in $O(n^2 \log(n/\varepsilon))$ rounds. However, our analysis of the convergence rate is more general and can be applied under the weaker assumption that $\|A\| < 1$. Then, the convergence time depends on $1 - \|A\|$ (see also Lemma 4.1 and Theorem 4.1). We usually refer to matrices similar to A, i.e., with infinity norm less than 1 and 0s in their diagonal, as *influence* matrices, and to matrices similar to B, i.e., to diagonal matrices with positive elements, as *self-confidence* matrices.

4.3.2 Average-Oriented Opinion Formation with Outdated Information

We study opinion formation when the agents have outdated information about the average public opinion. We assume an infinite increasing sequence of rounds $0 = \tau_0 < \tau_1 < \tau_2 < \cdots$ that describes an *update schedule* for the average opinion. At the end of round τ_p, the average $\mathsf{avg}(x(\tau_p))$ is announced to the agents. We refer to the rounds between two updates as an *epoch*. Specifically, the rounds $\tau_p + 1, \ldots, \tau_{p+1}$ comprise epoch p. We assume that the length of each epoch p, denoted by $k_p = \tau_{p+1} - \tau_p \geq 1$, is finite. The update schedule is the same for all agents, but the agents do not need to have any information about it. They only need to be aware of the most recent value of the average public opinion provided to them.

We now need to distinguish in Equations (4.2) and (4.3) between the influence from social neighbors, for which the most recent opinions $x(t-1)$ are used, and the influence from the average public opinion, where possibly outdated information is used. As such, we now rely on three different $n \times n$ matrices D, E and B. Self-confidence matrix B is defined as before. Influence matrix D has $d_{ii} = 0$, for all $i \in N$, and $d_{ij} = w_{ij}/S_i$, for all $j \neq i$, and accounts for the influence from social neighbors. Influence matrix E has

$e_{ii} = 0$, for all $i \in N$, and $e_{ij} = -\alpha_i/(n^2 S_i)$, for all $j \neq i$, and accounts for the influence from the average public opinion. By definition, $A = D + E$. Moreover, $\|D\| \leq 1 - 1/n$ and that $\|E\| \leq (n-1)/n^2$.

At the beginning of the opinion formation process, $x(0) = s$. For each round t in epoch p, $\tau_p + 1 \leq t \leq \tau_{p+1}$, the agent opinions are updated according to:

$$x(t) = Dx(t-1) + Ex(\tau_p) + Bs \tag{4.6}$$

We note that at the beginning of each epoch p, every agent i can subtract $x_i(\tau_p)$ from $n\,\mathsf{avg}(x(\tau_p))$ and compute $Ex(\tau_p)$, which is required in (4.6), as $-\frac{\alpha_i}{n^2 S_i}(n\,\mathsf{avg}(x(\tau_p)) - x_i(\tau_p))$.

4.3.3 Opinion Formation with Negative Influence

An interesting aspect of average-oriented games is that the influence matrix A may contain negative elements. Motivated by this observation, we prove our convergence results for a general domain of opinion formation games that may have negative weights w_{ij}. Similar to [17, 81], the individual cost function of each agent i is $C_i(x) = \sum_{j \neq i} w_{ij}(x_i - x_j)^2 + w_i(x_i - s_i)^2$ and i's best response to x_{-i} is

$$x_i = \frac{\sum_{j \neq i} w_{ij} x_j + w_i s_i}{w_i + \sum_{j \neq i} w_{ij}} \;. \tag{4.7}$$

The important difference is that now some w_{ij} may be negative. We require that for each agent i, $w_i > 0$ and $S_i = w_i + \sum_{j \neq i} w_{ij} > 0$ (and thus, $C_i(x)$ is strictly convex in x_i). The matrices A and B are defined as before. Namely, $a_{ij} = w_{ij}/S_i$, for all $i \neq j$, and B has $b_{ii} = w_i/S_i$ for all i. We always require that $\|A\| < 1 - \beta$, for some $\beta > 0$ (β may depend on n). Simultaneous best-response dynamics is again defined by (4.3).

4.4 Convergence of Average-Oriented Opinion Formation

For any nonnegative influence matrix A with $\|A\| \leq 1 - \beta$, (4.3) converges to the equilibrium point $x^* = (\mathbb{I} - A)^{-1} Bs$ within distance ε in $O(\log(\frac{\|B\|}{\varepsilon\beta})/\beta)$ rounds, as shown in [81, Lemma 3]. The following lemma shows that the same convergence rate holds for average-oriented opinion formation games, where A may contain negative elements. The proof is very similar to the proof of [81, Lemma 3] and we include it for completeness. The only minor difference is that the proof of Lemma 4.1 uses the infinity norm of A, instead of the largest eigenvalue of A in [81]. This allows for a direct generalization of Lemma 4.1 to the case of average-oriented opinion formation games with outdated information.

Lemma 4.1. *Let A be any influence matrix, possibly with negative elements, such that $\|A\| \leq 1-\beta$, for some $\beta > 0$. Then, for any self-confidence matrix B, any $s \in [0,1]^n$ and any $\varepsilon > 0$, the opinion formation process $x(t) = Ax(t-1) + Bs$ converges to $x^* = (\mathbb{I} - A)^{-1}Bs$ within distance ε in $O(\log(\frac{\|B\|}{\varepsilon\beta})/\beta)$ rounds.*

Proof. By (4.4), we have that for any $t \geq 1$, $x(t) = A^t s + \sum_{\ell=0}^{t-1} A^\ell Bs$. Since $\|A\| \leq 1-\beta$, $\|A^t\| \leq (1-\beta)^t$. Therefore, $\lim_{t\to\infty} A^t s = \vec{0}$ and (4.4) converges to $x^* = \sum_{\ell=0}^{\infty} A^\ell Bs$. Using the identity $\sum_{\ell=0}^{\infty} A^\ell = (\mathbb{I}-A)^{-1}$, we obtain that $x^* = (\mathbb{I}-A)^{-1}Bs$. We note that since $\|A\| < 1$, the matrix $\mathbb{I} - A$ is strictly diagonally dominant and thus non-singular. Moreover,

$$\|(\mathbb{I}-A)^{-1}\| \leq \sum_{\ell=0}^{\infty} \|A^\ell\| \leq \sum_{\ell=0}^{\infty} (1-\beta)^\ell = 1/\beta\,.$$

To bound the convergence time to x^*, we define $e(t) = \|x(t) - x^*\| = \max_{i \in N} |x_i(t) - x_i^*|$ as the distance of the opinions at time t to equilibrium. We next show that $e(t)$ is decreasing in t and obtain an upper bound on $t^*(\varepsilon) = \min\{t : e(t) \leq \varepsilon\}$. We observe that for any $t \geq 1$,

$$\begin{aligned} e(t) &= \|x(t) - x^*\| \\ &= \|Ax(t-1) + Bs - Ax^* - Bs\| \\ &\leq \|A\|\,\|x(t-1) - x^*\| \\ &\leq (1-\beta)e(t-1) \leq (1-\beta)^t e(0)\,. \end{aligned}$$

Since $s \in [0,1]^n$ and $\|(\mathbb{I}-A)^{-1}\| \leq 1/\beta$, we obtain that

$$\|x^*\| \leq \|(\mathbb{I}-A)^{-1}Bs\| \leq \|(\mathbb{I}-A)^{-1}\|\,\|B\|\,\|s\| \leq \|B\|/\beta\,.$$

Since $x(0) = s$, we have that $e(0) = \|s - x^*\| \leq 1 + \|B\|/\beta$. Hence, $t^*(\varepsilon) = O(\log(\frac{\|B\|}{\varepsilon\beta})/\beta)$. □

Since $\mathbb{I} - A$ is nonsingular, x^* is the unique opinion vector that satisfies $x^* = Ax^* + Bs$. Thus, x^* is the unique equilibrium of the corresponding opinion formation game. Moreover, since for average-oriented games $\|A\| \leq 1 - 2/n^2$, Lemma 4.1 implies the following:

Corollary 4.1. *Any average-oriented game satisfying Assumption 1 admits a unique equilibrium $x^* = (\mathbb{I}-A)^{-1}Bs$, and for any $\varepsilon > 0$, (4.3) converges to x^* within distance ε in $O(n^2 \log(n/\varepsilon))$ rounds.*

4.4.1 Convergence with Outdated Information

Next, we extend Lemma 4.1 to the case where the agents use possibly outdated information about the average public opinion in each round. More generally, we establish convergence for a general domain with negative influence between the agents, which includes average-oriented opinion formation processes as a special case.

Theorem 4.1. *Let D and E be influence matrices, possibly with negative elements, such that $\|D\| \leq 1-\beta_1$, $\|E\| \leq 1-\beta_2$, for some $\beta_1, \beta_2 \in (0,1)$ with $\beta_1+\beta_2 > 1$. Then, for any self-confidence matrix B, any $s \in [0,1]^n$, any update schedule $0 = \tau_0 < \tau_1 < \tau_2 < \cdots$ and any $\varepsilon > 0$, the opinion formation process (4.6) converges to $x^* = (\mathbb{I} - (D+E))^{-1}Bs$ within distance ε in $O(\log(\frac{\|B\|}{\varepsilon\beta})/\beta)$ epochs, where $\beta = \beta_1+\beta_2-1 > 0$.*

Proof. We observe that $x^* = (\mathbb{I} - (D+E))^{-1}Bs$ is the unique solution of $x^* = Dx^* + Ex^* + Bs$ (as in Lemma 4.1, since $\|E+D\| \leq 1-\beta$, with $\beta > 0$, the matrix $\mathbb{I} - (D+E)$ is non-singular). Hence, if (4.6) converges, it converges to x^*. To show convergence, we bound the distance of $x(t)$ to x^* by a decreasing function of t and show an upper bound on $t^*(\varepsilon) = \min\{t : e(t) \leq \varepsilon\}$.

As in the proof of Lemma 4.1, for each round $t \geq 1$, we define $e(t) = \|x(t) - x^*\|$ as the distance of the opinions at time t to x^*. For convenience, we also define

$$f(\beta_1, \beta_2, k) = (1-\beta_1)^k + (1-\beta_2)\frac{1-(1-\beta_1)^k}{\beta_1}\,.$$

For any fixed value of $\beta_1, \beta_2 \in (0,1)$ with $\beta_1+\beta_2 > 1$, $f(\beta_1,\beta_2,k)$ is a decreasing function of k. Actually, the derivative of f with respect to k is equal to $\log(1-\beta_1)(1-\beta_1)^k(1-\frac{1-\beta_2}{\beta_1})$, which is negative, because $1 > (1-\beta_2)/\beta_1$, since $\beta_1+\beta_2 > 1$.

We next show that:

Claim (i). For any epoch $p \geq 0$ and any round k, $0 \leq k \leq k_p$, in epoch p,

$$e(\tau_p + k) \leq f(\beta_1, \beta_2, k)e(\tau_p)\,.$$

Claim (ii). In the last round $\tau_{p+1} = \tau_p + k_p$ of each epoch $p \geq 0$, $e(\tau_{p+1}) \leq (1-\beta)e(\tau_p)$.

Claim (i) shows that the distance to equilibrium decreases from each round to the next within each epoch, while Claim (ii) shows that the distance to equilibrium decreases geometrically from the last round of each epoch to the last round of the next epoch. Combining Claim (i) and Claim (ii), we obtain that for any epoch $p \geq 0$ and any round k, $0 \leq k \leq k_p$, in epoch p, $e(\tau_p + k) \leq f(\beta_1, \beta_2, k)(1-\beta)^p e(0)$. Therefore, for any update schedule $\tau_0 < \tau_1 < \tau_2 < \cdots$, the opinion formation process (4.6) converges to $(\mathbb{I} - (D+E))^{-1}Bs$ in $O(\log(e(0)/\varepsilon)/\beta)$ epochs.

To prove Claim (i), we fix any epoch $p \geq 0$ and apply induction on k. The basis, where $k = 0$, holds because $f(\beta_1, \beta_2, 0) = 1$. For any round k, with $1 \leq k \leq k_p$, in p, we have that:

$$\begin{aligned} e(\tau_p + k) &= \|Dx(\tau_p + k - 1) + Ex(\tau_p) + Bs - (Dx^* + Ex^* + Bs)\| \\ &\leq \|D\| \, \|x(\tau_p + k - 1) - x^*\| + \|E\| \, \|x(\tau_p) - x^*\| \\ &\leq (1-\beta_1)e(\tau_p + k - 1) + (1-\beta_2)e(\tau_p) \\ &\leq (1-\beta_1)f(\beta_1, \beta_2, k-1)e(\tau_p) + (1-\beta_2)e(\tau_p) = f(\beta_1, \beta_2, k)e(\tau_p)\,. \end{aligned}$$

The first inequality follows from the properties of matrix norms. The second inequality holds because $\|D\| \leq 1-\beta_1$ and $\|E\| \leq 1-\beta_2$. The third inequality follows from the induction hypothesis. Finally, we use that for any $k \geq 1$, $(1-\beta_1)f(\beta_1, \beta_2, k-1) + 1 - \beta_2 = f(\beta_1, \beta_2, k)$.

To prove Claim (ii), we fix any epoch $p \geq 0$ and apply claim (i) to the last round $\tau_{p+1} = \tau_p + k_p$, with $k_p \geq 1$, of epoch p. Hence, $e(\tau_{p+1}) = \|x(\tau_p + k_p) - x^*\| \leq f(\beta_1, \beta_2, k_p)e(\tau_p)$.

We next show that $f(\beta_1, \beta_2, k_p) \leq 2 - (\beta_1 + \beta_2) = 1 - \beta$, which concludes the proof of the claim. The inequality holds because for any integer $k \geq 1$, $f(\beta_1, \beta_2, k)$ is a convex function of β_1. For a formal proof, we fix any $k \geq 1$ and any $\beta_2 \in (0,1)$, and consider the functions $g(x) = (1-x)^k + \frac{1-(1-x)^k}{x}(1-\beta_2)$ and $h(x) = 2 - \beta_2 - x$, where $x \in [1-\beta_2, 1]$ (since we assume that $\beta_1 \in (0,1)$ and that $\beta_1 > 1 - \beta_2$). For any fixed value of $\beta_2 \in (0,1)$, $h(x)$ is a linear function of x with $h(1-\beta_2) = 1$ and $h(1) = 1 - \beta_2$. For any fixed value of $k \geq 1$ and $\beta_2 \in (0,1)$, $g(x)$ is a convex function of x with $g(1-\beta_2) = 1 = h(1-\beta_2)$ and $g(1) = 1 - \beta_2 = h(1)$. Therefore, for any $\beta_1 \in [1-\beta_2, 1]$, $g(\beta_1) \leq h(\beta_1) = 2 - (\beta_1 + \beta_2)$.

To obtain an upper bound on $e(0) = \|s - x^*\|$, we work as in the proof of Lemma 4.1, using the fact that $\|D + E\| \leq 1 - \beta$, and show first that $\|(\mathbb{I} - (D+E))^{-1}\| \leq 1/\beta$ and then that $\|x^*\| \leq \|B\|/\beta$. Since $x(0) = s$, we have that $e(0) = \|s - x^*\| \leq 1 + \|B\|/\beta$. Using the fact that for each epoch $p \geq 0$ and for every round k, $0 \leq k \leq k_p$, in p, $e(\tau_p + k) \leq f(\beta_1, \beta_2, k)(1-\beta)^p e(0)$, we obtain that $t^*(\varepsilon) = O(\log(\frac{\|B\|}{\varepsilon\beta})/\beta)$ epochs. □

For average-oriented opinion formation games, $D + E = A$, $\|D\| \leq 1 - 1/n$ and $\|E\| \leq (n-1)/n^2$. Hence, applying Theorem 4.1 with $\beta \geq 1/n^2$, we obtain the following:

Corollary 4.2. *For any update schedule and any* $\varepsilon > 0$*, the opinion formation process* (4.6) *with outdated information about* $\mathsf{avg}(x(t))$ *converges to the equilibrium* $x^* = (\mathbb{I} - A)^{-1}Bs$ *of the corresponding average-oriented game within distance* ε *in* $O(n^2 \log(n/\varepsilon))$ *epochs.*

4.5 The Price of Anarchy of Symmetric Average-Oriented Games

In this section we proceed to bound the PoA of average-oriented opinion formation games. We concentrate on the most interesting case of symmetric games, since nonsymmetric opinion formation games can have a PoA of $\Omega(n)$, even if $\alpha = 0$ [17]. Recall that for symmetric games, $w_{ij} = w_{ji}$ for all agent pairs i, j, and $w_i = 1$ and $\alpha_i = \alpha$, for all agents i.

Our analysis generalizes a local smoothness argument put forward in [15]. Such arguments have been extensively used in the algorithmic game theory literature to provide upper bounds on the Price of Anarchy and they are based on the notion of (λ, μ)-*locally smooth* introduced in [128].

Definition 4.4. *[128] A game is* (λ, μ)-locally smooth *if there exist* $\lambda > 0$ *and* $\mu \in (0, 1)$*, such that for all* $x, z \in \mathbf{R}^n$

$$\sum_{i \in N} C_i(x) + \sum_{i \in N} (z_i - x_i) \frac{dC_i(x)}{dx_i} \leq \lambda \sum_{i \in N} C_i(z) + \mu \sum_{i \in N}^{n} C_i(x) \tag{4.8}$$

Proposition 4.1. *If a game is* $(\lambda, \mu)-$*locally smooth, then*

$$PoA \ \leq \frac{\lambda}{1 - \mu}$$

Proof. Let x^* the Nash Equilibrium of the game and o^* the opinion vector minimizing the total disagreement cost $\sum_{i \in N} C_i(x)$. Since x^* is the Nash Equilibrium of the game, $\frac{dC_i(x^*)}{dx_i} = 0$ for each agent i. Hence, applying (4.8) for $x = x^*$ and $z = o^*$, we obtain that $\mathsf{PoA} \leq \lambda/(1 - \mu)$. □

For symmetric games without aggregation ($\alpha = 0$), it is known [15] that for any $s \in [0, 1]^n$, the game is $(3/4, 1/4)$-locally smooth and thus the PoA of symmetric opinion formation games without aggregation can be bounded to at most $9/8$ [15]. This is tight as shown in [17].

This elegant approach cannot be directly generalized to symmetric average-oriented opinion formation games, because the function $\sum_{i=1}^{n} (\mathsf{avg}(x) - s_i)^2$ is not (λ, μ)-locally smooth for any $\mu < 1$. To circumvent this difficulty, we use the local smoothness technique in a more creative way. Observe that finding appropriate values of λ, μ that satisfy (4.8) for all $x, z \in [0, 1]^n$ may be both a hard and a redundant task, because (4.8) is applied only for $x = x^*$ and $z = o^*$, where x^* denotes the Nash equilibrium and o^* denotes the optimal vector. Next, we derive appropriate values of λ, μ so that (4.8) holds for all opinion vectors $x, z \in [0, 1]^n$ for which $\mathsf{avg}(x) = \mathsf{avg}(s)$. In Proposition 4.2, we show that for symmetric opinion formation games, the average equilibrium opinion is equal to the average belief, which allows us to bound the PoA.

Proposition 4.2. *Let x^* be the equilibrium and s the internal opinion vector of any symmetric average-oriented opinion formation game. Then,*

$$\mathsf{avg}(x^*) = \mathsf{avg}(s)$$

Proof. The following holds for the opinion x_i^* of any agent i at Nash equilibrium x^*:

$$x_i^* + x_i^* \sum_{j \neq i} w_{ij} = (1 + \alpha/n)s_i + \sum_{j \neq i} w_{ij} x_j^* - (\alpha/n)\mathsf{avg}(x^*) \ .$$

By summing up these inequalities for all agents $i \in [n]$,

$$n\,\mathsf{avg}(x^*) + \sum_{i \in N} x_i^* \sum_{j \neq i} w_{ij} = (n + \alpha)\mathsf{avg}(s) + \sum_{i \in N} \sum_{j \neq i} w_{ij} x_j^* - \alpha\,\mathsf{avg}(x^*) \ .$$

Since the game is symmetric with $w_{ij} = w_{ji}$ for all $i \neq j$,

$$\sum_{i \in N} x_i^* \sum_{j \neq i} w_{ij} = \sum_{i \in N} \sum_{j \neq i} w_{ij} x_j^* = \sum_{i,j: i<j} w_{ij}(x_i^* + x_j^*) \ .$$

Therefore, we obtain that at the equilibrium x^*, $(n+\alpha)\mathsf{avg}(x^*) = (n+\alpha)\mathsf{avg}(s)$, which directly implies the proposition. □

In the analysis of PoA, we use the following technical proposition repeatedly.

Proposition 4.3. *For any $\gamma, \lambda, \mu \geq 0$ and $z, x \in \mathbf{R}$ such that $\lambda\mu \geq \gamma^2$,*

$$2\gamma z x \leq \lambda z^2 + \mu x^2$$

Proof. The claim holds trivially if $zx < 0$. In case where $zx \geq 0$, the claim follows from:

$$0 \leq (\sqrt{\lambda} z - \sqrt{\mu} x)^2 = \lambda z^2 + \mu x^2 - 2\sqrt{\lambda\mu} z x \leq \lambda z^2 + \mu x^2 - 2\gamma z x$$

The last inequality holds because $\lambda\mu \geq \gamma^2$ implies that $-\sqrt{\lambda\mu} \leq -\gamma$. □

Based on these properties, we show that the **PoA** of symmetric average-oriented games tends to $9/8$, which is the **PoA** of symmetric opinion formation games without aggregation. The proof is based on the following technical (and more general) lemma:

Lemma 4.2. *Let $\mathcal{G}$ be any symmetric average-oriented opinion formation game with n agents, agent belief vector s and influence $\alpha \geq 0$. Then, for all $x, z \in \mathbf{R}^n$ such that* $\text{avg}(x) = \text{avg}(s)$*,*

$$\sum_{i \in N} C_i(x) + \sum_{i \in N} (z_i - x_i) \frac{dC_i(x)}{dx_i} \leq \nu_1 \sum_{i \in N} C_i(z) + \nu_2 \sum_{i \in N} C_i(x)$$

where $\nu_1 = \max\{3/4 + \mu, \delta\}$ and $\nu_2 = \max\{1/3 + \mu, 1 - \delta + 2\lambda\}$, for all $\lambda > 0$ and $\mu \in (0, 1)$ such that $\lambda\mu \geq \alpha/n^2$ and for all $\delta > 0$.

Proof. We recall that the individual cost of each agent i with respect to opinions x is

$$C_i(x) = \sum_{i \neq j} w_{ij}(x_i - x_j)^2 + (x_i - s_i)^2 + \alpha(\text{avg}(x) - s_i)^2$$

and that the social cost is $C(x) = \sum_{i \in N} C_i(x)$. We divide agent's i personal cost $C_i(x)$ into three parts $C_i(x) = F_i(x) + I_i(x) + A_i(x)$, where $F_i(x) = \sum_{j \neq i} w_{ij}(x_i - x_j)^2$, $I_i(x) = (x_i - s_i)^2$ and $A_i(x) = \alpha(\text{avg}(x) - s_i)^2$.

Following this notation, we have that:

$$\begin{aligned}
F(x) &= \sum_{i \in N} F_i(x) = \sum_{i \in N} \sum_{j \neq i} w_{ij}(x_i - x_j)^2 = 2 \sum_{i,j: i<j} w_{ij}(x_i - x_j)^2 \\
I(x) &= \sum_{i \in N} I_i(x) = \sum_{i \in N} (x_i - s_i)^2 = (x - s)^T (x - s) \\
A(x) &= \sum_{i \in N} A_i(x) = \alpha \sum_{i \in N} (\text{avg}(x) - s_i)^2 = \alpha(\text{avg}(x) - s)^T (\text{avg}(x) - s) \ .
\end{aligned}$$

Consequently, the social cost can be written as $C(x) = F(x) + I(x) + A(x)$. We introduce

$$\begin{aligned}
F'(x) &= \left(\frac{dF_1(x)}{dx_1}, \cdots, \frac{dF_n(x)}{dx_n} \right) \\
I'(x) &= \left(\frac{dI_1(x)}{dx_1}, \cdots, \frac{dI_n(x)}{dx_n} \right) \\
A'(x) &= \left(\frac{dA_1(x)}{dx_1}, \cdots, \frac{dA_n(x)}{dx_n} \right)
\end{aligned}$$

We observe that $A'(x) = (2\alpha/n)(\mathbf{avg}(x) - s)$. For simplicity and brevity, here and in the proof of Theorem 4.4, we slightly abuse the notation by letting $\mathbf{avg}(x)$ denote a vector with all its coordinates equal to $\mathbf{avg}(x)$. The following two propositions are proven in [15, Sec. 3.1] for more general cost functions. We provide their proofs here, for the sake of completeness.

Proposition 4.4 ([15]). *For any symmetric matrix $W = (w_{ij})$, any $x, z \in \mathbf{R}^n$, and any $\lambda > 0$ and $\mu \in (0, 1)$ with $\lambda \geq 1/(4\mu)$,*

$$F(x) + (z - x)^T F'(x) \leq \lambda F(z) + \mu F(x)$$

Proof. To establish the proposition, we consider each agent pair i, j, with $i \neq j$, separately. Since for any agent pair i, j, $w_{ij} = w_{ji}$, we have that for any $\lambda > 0$ and $\mu \in (0, 1)$ with $\lambda\mu \geq 1/4$,

$$\begin{aligned} F(x) + (z - x)^T F'(x) &= 2 \sum_{i,j:i \neq j} w_{ij}((x_i - x_j)^2 + (z_i - x_i)(x_i - x_j) + (z_j - x_j)(x_j - x_i)) \\ &= 2 \sum_{i,j:i \neq j} w_{ij}((x_i - x_j)^2 + (z_i - z_j)(x_i - x_j) - (x_i - x_j)^2) \\ &= 2 \sum_{i,j:i \neq j} w_{ij}(z_i - z_j)(x_i - x_j) \\ &\leq 2\lambda \sum_{i,j:i \neq j} w_{ij}(z_i - z_j)^2 + 2\mu \sum_{i,j:i \neq j} w_{ij}(x_i - x_j)^2 \\ &= \lambda F(z) + \mu F(x)\,. \end{aligned}$$

For the inequality, we apply Proposition 4.3 with $\gamma = 1/2$. Therefore, for any $z_i, z_j, x_i, x_j \in \mathbf{R}$ and any $\lambda, \mu > 0$ with $\lambda\mu \geq 1/4$, $(z_i - z_j)(x_i - x_j) \leq \lambda(z_i - z_j)^2 + \mu(x_i - x_j)^2$. □

Proposition 4.5 ([15]). *For any $x, z, s \in \mathbf{R}^n$, $\lambda > 0$ and $\mu \in (0, 1)$ with $\lambda \geq 1/(\mu + 1)$,*

$$I(x) + (z - x)^T I'(x) \leq \lambda I(z) + \mu I(x)$$

Proof. To establish the proposition, we consider each agent i separately. We have that for any $\lambda > 0$ and $\mu \in (0, 1)$ such that $\lambda(\mu + 1) \geq 1$,

$$\begin{aligned}
I(x) + (z - x)^T I'(x) &= \sum_{i \in N} ((x_i - s_i)^2 + 2(z_i - x_i)(x_i - s_i)) \\
&= \sum_{i \in N} ((x_i - s_i)^2 + 2(z_i - s_i)(x_i - s_i) + 2(s_i - x_i)(x_i - s_i)) \\
&= \sum_{i \in N} ((x_i - s_i)^2 + 2(z_i - s_i)(x_i - s_i) - 2(x_i - s_i)^2) \\
&= \sum_{i \in N} (2(z_i - s_i)(x_i - s_i) - (x_i - s_i)^2) \\
&\leq \lambda \sum_{i \in N} (z_i - s_i)^2 + \mu \sum_{i \in N} (x_i - s_i)^2 \\
&= \lambda I(z) + \mu I(x)
\end{aligned}$$

For the inequality, we apply Proposition 4.3 with $\gamma = 1$ and $\mu + 1$ instead of μ. Thus, we obtain that for any $x_i, z_i, s_i \in \mathbf{R}$ and for any $\lambda > 0$ and $\mu \in (0, 1)$ such that $\lambda(\mu + 1) \geq 1$, $2(z_i - s_i)(x_i - s_i) \leq \lambda(z_i - s_i)^2 + (\mu + 1)(x_i - s_i)^2$, which implies the inequality above. □

Next, using Proposition 4.2, we obtain a similar upper bound on $A(x) + (z - x)^T A'(x)$.

Proposition 4.6. *For any $\alpha > 0$, any $x, z, s \in \mathbf{R}^n$ with $\mathrm{avg}(x) = \mathrm{avg}(s)$, any $\delta \geq 0$, and any $\lambda > 0$ and $\mu \in (0, 1)$ such that $\lambda\mu \geq \alpha/n^2$,*

$$A(x) + (z - x)^T A'(x) \leq \delta A(z) + \mu I(z) + (1 - \delta + 2\lambda)A(x) + \mu I(x) \ . \quad (4.9)$$

Proof. Applying first-order optimality conditions, we obtain that any vector $x \in \mathbf{R}^n$ with $\mathrm{avg}(x) = \mathrm{avg}(s)$ minimizes $A(x)$. Therefore, for any $x \in \mathbf{R}^n$, $A(x) \leq A(x)$, and for any $\delta \geq 0$, $A(x) \leq \delta A(x) + (1 - \delta)A(x)$.

To complete the proof of (4.9), we observe that for any $\lambda > 0$, $\mu \in (0, 1)$ with $\lambda\mu \geq \alpha/n^2$,

$$\begin{aligned}
(z - x)^T A'(x) &= \sum_{i \in N} (2\alpha/n)(z_i - x_i)(\mathrm{avg}(x) - s_i) \\
&= \sum_{i \in N} ((2\alpha/n)(z_i - s_i)(\mathrm{avg}(x) - s_i) + (2\alpha/n)(s_i - x_i)(\mathrm{avg}(x) - s_i)) \\
&\leq \sum_{i \in N} (2\lambda\alpha(\mathrm{avg}(x) - s_i)^2 + \mu(z_i - s_i)^2 + \mu(x_i - s_i)^2) \\
&= 2\lambda A(x) + \mu I(z) + \mu I(x).
\end{aligned}$$

For the inequality, we apply Proposition 4.3, with $\gamma = \sqrt{\alpha}/n$, to $(2\alpha/n)(z_i - s_i)(\mathrm{avg}(x) - s_i)$ and to $(2\alpha/n)(s_i - x_i)(\mathrm{avg}(x) - s_i)$. Hence, we obtain that for any $\lambda > 0$ and $\mu \in (0, 1)$ such that $\lambda\mu \geq \alpha/n^2$, $(2\alpha/n)(z_i - s_i)(\mathsf{avg}(x) - s_i) \leq \mu(z_i - s_i)^2 + \lambda\alpha(\mathsf{avg}(x) - s_i)^2$ and $(2\alpha/n)(s_i - x_i)(\mathsf{avg}(x) - s_i) \leq \mu(x_i - s_i)^2 + \lambda\alpha(\mathsf{avg}(x) - s_i)^2$. □

Applying Propositions 4.4 and 4.5 with $\lambda = 3/4$ and $\mu = 1/3$, and using (4.9), we obtain that for any $\delta \geq 0$ and for any $\lambda > 0$ and $\mu \in (0, 1)$ such that $\lambda\mu \geq \alpha/n^2$,

$$\begin{aligned} C(x) + (z - x)^T C'(x) &\leq \tfrac{3}{4}F(z) + \left(\tfrac{3}{4} + \mu\right) I(x) + \delta A(x) + \tfrac{1}{3}F(x) + \\ &\quad \left(\tfrac{1}{3} + \mu\right) I(z) + (1 - \delta + 2\lambda)A(z) \\ &\leq \nu_1 C(z) + \nu_2 C(x)\,, \end{aligned}$$

where $\nu_1 = \max\{3/4 + \mu, \delta\}$ and $\nu_2 = \max\{1/3 + \mu, 1 - \delta + 2\lambda\}$. □

The main result of this section is an immediate consequence of Lemma 4.2.

Theorem 4.2. *Let $\mathcal{G}$ be any symmetric average-oriented opinion formation game with n agents and influence $\alpha \geq 0$. Then,* $\mathsf{PoA}(\mathcal{G}) \leq 9/8 + O(\alpha/n^2)$.

Proof. Let x^* be the Nash equilibrium and let o^* be the optimal solution. By Proposition 4.2, $\mathsf{avg}(x^*) = \mathsf{avg}(s)$. Therefore, Lemma 4.2 implies that

$$C(x^*) + (o^* - x^*)^T C'(x) \leq \nu_1 C(o^*) + \nu_2 C(x^*)$$

where $\nu_1 = \max\{3/4 + \mu, \delta\}$ and $\nu_2 = \max\{1/3 + \mu, 1 - \delta + 2\lambda\}$, for all $\lambda > 0$ and $\mu \in (0, 1)$ such that $\lambda\mu \geq \alpha/n^2$ and for all $\delta > 0$. Since x^* is an equilibrium, $C'(x^*) = \vec{0}$. Hence, for all $\nu_2 \in (0, 1)$, $\mathsf{PoA}(\mathcal{G}) \leq \nu_1/(1 - \nu_2)$, or equivalently,

$$\mathsf{PoA}(\mathcal{G}) \leq \frac{\max\{3/4 + \mu, \delta\}}{1 - \max\{1/3 + \mu, 1 - \delta + 2\lambda\}} \tag{4.10}$$

If α/n^2 is small enough, e.g., if $\alpha/n^2 \leq 1/2400$, we use $\delta = 3/4$, $\lambda = 1/24$ and $\mu = 24\alpha/n^2$ in (4.10) and obtain that $\mathsf{PoA}(\mathcal{G}) \leq 9/8 + O(\frac{\alpha}{n^2})$. Otherwise, we use $\mu = 1/3$, $\lambda = 3\alpha/n^2$ and $\delta = 6\alpha/n^2 + 1/3$, and obtain that $\mathsf{PoA}(\mathcal{G}) = O(\frac{\alpha}{n^2})$. □

4.6 Average-Oriented Games with Restricted Opinions

A frequent assumption in the literature on opinion formation is that agent beliefs come from a finite interval of nonnegative real numbers. Then, by scaling we can assume beliefs $s_i \in [0, 1]$. If the influence matrix A is nonnegative, then since $b_{ii} + \sum_{j=1}^{n} a_{ij} = 1$ for all $i \in N$, we have that the equilibrium opinions are $x^* = (\mathbb{I} - A)^{-1} Bs \in [0, 1]^n$. In contrast, for the more general domain we treat here, an important side-effect of negative influence is that the best-response (and equilibrium) opinions may not belong to $[0, 1]$. Motivated by this observation, we consider a *restricted* variant of opinion formation games, where the (best-response and equilibrium) opinions are restricted to $[0, 1]$. We strive to understand how this restriction of public opinions to $[0, 1]$ affects the convergence properties and the price of anarchy of average-oriented games.

To distinguish restricted opinion formation processes from their unrestricted counterparts, we use $y(t)$ to denote the opinion vectors restricted to $[0, 1]^n$. For restricted average-oriented games and restricted games with negative influence, the best-response opinion y_i of each agent i to y_{-i} is again computed by (4.2) and (4.7), respectively. But now, if the resulting value is $y_i < 0$, we increase it to $y_i = 0$, while if $y_i > 1$, we decrease it to $y_i = 1$. Since the individual cost $C_i(y)$ is a strictly convex function of y_i, the restriction of y_i to $[0, 1]$ results in a minimizer $y^* \in [0, 1]$ of $C_i(y, y_{-i})$.

Similarly, the restricted opinion formation process is described by

$$y(t) = [Ay(t-1) + Bs\,]_{[0,1]}\,, \tag{4.11}$$

where $[\cdot]_{[0,1]}$ denotes the restriction of public opinions $y(t)$ to $[0, 1]^n$ described above. The influence matrix A (and the influence matrices D and E for processes with outdated information) and the self-confidence matrix B are computed as for standard (or unrestricted) opinion formation processes.

4.6.1 Convergence of Restricted Opinion Formation Processes

We show results for restricted opinion formation processes that are equivalent to Lemma 4.1 and Theorem 4.1. As in Section 4.4, we prove our results for the more general setting of negative influence. Using Lemma 4.3 and Theorem 4.3, it is straightforward to obtain the results of Corollary 4.1 and Corollary 4.2 also for restricted average-oriented processes.

Lemma 4.3. *Let A be any influence matrix, possibly with negative elements, such that $\|A\| \le 1 - \beta$, for some $\beta > 0$. Then, for any self-confidence matrix B, any $s \in [0, 1]^n$ and any $\varepsilon > 0$, the opinion formation process*

$$y(t) = [Ay(t-1) + Bs\,]_{[0,1]}$$

admits a unique equilibrium y^* and converges to it within distance ε in $O(\log(\frac{1}{\varepsilon})/\beta)$ rounds.

Proof. In the restricted opinion formation game, the agent opinions lie in the convex set $[0, 1]$. The individual cost $C_i(y)$ of each agent i is a continuous function of y and strictly convex in y_i. Hence, according to the results of [126], the restricted game admits a unique equilibrium y^* which satisfies $y^* = [Ay^* + Bs\,]_{[0,1]}$. Specifically, the existence of an equilibrium y^* follows from [126], since the restricted opinion formation game is a convex game. The uniqueness of y^* follows from [126] and from the fact that the function $\sum_{i\in N} C_i(y)$ is diagonally strictly convex. The latter holds because the symmetric matrix obtained by adding $2B$ to the Laplacian of $A + A^T$ is positive definite.

Next we bound the convergence time to y^* as in the proof of Lemma 4.1. For any $t \geq 1$, we define $e(t) = \|y(t) - y^*\|$ as the distance of the opinions at time t to equilibrium. We observe that for any round $t \geq 1$,

$$\begin{aligned} e(t) = \|y(t) - y^*\| &\leq \|Ay(t-1) + Bs - Ay^* - Bs\| \\ &\leq \|A\|\,\|y(t-1) - y^*\| \leq (1-\beta)e(t-1) \leq (1-\beta)^t e(0)\,. \end{aligned}$$

For the first inequality, we recall that $y(t)$ (resp. y^*) is obtained by computing $Ay(t-1) + Bs$ (resp. $Ay^* + Bs$) and then restricting any negative opinions to 0 and any opinions larger than 1 to 1. By a straightforward inspection of all possible 9 cases depending on whether $y_i(t)$ and y_i^* are negative, in $[0, 1]$ or greater than 1, we conclude that opinion restriction to $[0, 1]$ does not increase $|y_i(t) - y_i^*|$ for any i. Since $y(0) = s \in [0, 1]^n$ and $y^* \in [0, 1]^n$, $e(0) \leq 1$. Hence, after $t^*(\varepsilon) = O(\log(\frac{1}{\varepsilon})/\beta)$ rounds $y(t)$ is within distance ε to y^*. □

The proof of the following theorem is similar to the proof of Theorem 4.1.

Theorem 4.3. *Let D and E be influence matrices, possibly with negative elements, such that $\|D\| \leq 1 - \beta_1$, $\|E\| \leq 1 - \beta_2$, for some $\beta_1, \beta_2 \in (0, 1)$ with $\beta_1 + \beta_2 > 1$. Then, for any self-confidence matrix B, any $s \in [0, 1]^n$, any update schedule $0 = \tau_0 < \tau_1 < \tau_2 < \cdots$, the restricted opinion formation process*

$$y(t) = [Dy(t-1) + Ey(\tau_p) + Bs\,]_{[0,1]}$$

converges to the unique equilibrium point y^* of

$$y'(t) = [(D + E)y'(t-1) + Bs\,]_{[0,1]}$$

For any $\varepsilon > 0$, $y(t)$ is within distance ε to y^* after $O(\log(\frac{1}{\varepsilon})/\beta)$ epochs, where $\beta = \beta_1 + \beta_2 - 1$.

Proof. Lemma 4.3 shows that for the restricted opinion formation process $y'(t) = [(D+E)y'(t-1) + Bs]_{[0,1]}$, there is a unique equilibrium point $y^* \in [0,1]^n$ that satisfies $y^* = [(D+E)y^* + Bs]_{[0,1]}$. Provided that it exists, the equilibrium of the restricted opinion formation process with outdated information $y(t) = [Dy(t-1) + Ey(\tau_p) + Bs]_{[0,1]}$ must satisfy $y^* = [Dy^* + Ey^* + Bs]_{[0,1]}$, due to the existence of infinite update points where all agents have accurate information about the current public opinion vector. So, if the process with outdated information admits an equilibrium, it must be unique and equal to y^*. We next show that this is indeed the case, by bounding from above the distance of $y(t)$ to y^* by a decreasing function of t and by establishing an upper bound on the convergence time.

For every round $t \geq 1$, we define $e(t) = \|y(t) - y^*\|$ as the distance of the opinions at time t to y^*. We proceed similarly to the proof of Theorem 4.1. As before, we define

$$f(\beta_1, \beta_2, k) = (1-\beta_1)^k + (1-\beta_2)\frac{1-(1-\beta_1)^k}{\beta_1}\,.$$

We recall that for any fixed $\beta_1, \beta_2 \in (0,1)$ with $\beta_1 + \beta_2 > 1$, $f(\beta_1, \beta_2, k)$ is a decreasing function of k.

We next show that:

Claim (i). For every epoch $p \geq 0$ and every round k, $0 \leq k \leq k_p$, in epoch p,

$$e(\tau_p + k) \leq f(\beta_1, \beta_2, k)e(\tau_p)\,.$$

Claim (ii). In the last round $\tau_{p+1} = \tau_p + k_p$ of each epoch $p \geq 0$, $e(\tau_{p+1}) \leq (1-\beta)e(\tau_p)$.

Claims (i) and (ii) imply that for each epoch $p \geq 0$ and every round k, $0 \leq k \leq k_p$, in epoch p, $e(\tau_p + k) \leq f(\beta_1, \beta_2, k)(1-\beta)^p e(0)$. This immediately implies that for any update schedule $\tau_0 < \tau_1 < \tau_2 < \cdots$, the opinion formation process $y(t) = [Dy(t-1) + Ey(\tau_p) + Bs]_{[0,1]}$ converges to y^*. Moreover, since $e(0) = \|s - y^*\| \leq 1$, $y(t)$ is within distance ε to y^* in $O(\log(\frac{1}{\varepsilon})/\beta)$ epochs.

The proofs of Claim (i) and Claim (ii) are essentially identical to the proofs of the corresponding claims in the proof of Theorem 4.1. We include the details for completeness. To prove Claim (i), we fix an epoch $p \geq 0$ and apply induction on k. The basis, where $k = 0$, holds because $f(\beta_1, \beta_2, 0) = 1$.

For any round k, with $1 \leq k \leq k_p$, in p, we have that:

$$\begin{aligned}
e(\tau_p + k) &= \|y(\tau_p + k) - y^*\| \\
&= \|[Dy(\tau_p + k - 1) + Ey(\tau_p) + Bs]_{[0,1]} - [Dy^* + Ey^* + Bs]_{[0,1]}\| \\
&\leq \|(Dy(\tau_p + k - 1) + Ey(\tau_p) + Bs) - (Dy^* + Ey^* + Bs)\| \\
&\leq \|D\| \, \|y(\tau_p + k - 1) - y^*\| + \|E\| \, \|y(\tau_p) - y^*\| \\
&\leq (1 - \beta_1)e(\tau_p + k - 1) + (1 - \beta_2)e(\tau_p) \\
&\leq (1 - \beta_1)f(\beta_1, \beta_2, k - 1)e(\tau_p) + (1 - \beta_2)e(\tau_p) \\
&= f(\beta_1, \beta_2, k)e(\tau_p) \, .
\end{aligned}$$

For the first inequality, we use that opinion restriction to $[0, 1]$ does not increase $|y_i(t) - y_i^*|$ for any i, as it is explained in the proof of Lemma 4.3. The second inequality follows from the properties of matrix norms. The third inequality holds because $\|D\| \leq 1 - \beta_1$ and $\|E\| \leq 1 - \beta_2$. The fourth inequality follows from the induction hypothesis. Finally, we observe that for any integer $k \geq 1$, $(1 - \beta_1)f(\beta_1, \beta_2, k - 1) + 1 - \beta_2 = f(\beta_1, \beta_2, k)$.

To prove Claim (ii), we fix any epoch $p \geq 0$ and apply claim (i) to the last round $\tau_{p+1} = \tau_p + k_p$ of epoch p, where $k_p \geq 1$. Hence, we obtain that:

$$e(\tau_{p+1}) = \|y(\tau_p + k_p) - y^*\| \leq f(\beta_1, \beta_2, k_p)e(\tau_p) \leq (2 - \beta_1 - \beta_2)e(\tau_p) = (1 - \beta)e(\tau_p) \, ,$$

where $\beta = \beta_1 + \beta_2 - 1$. The last inequality follows from convexity and has already been proven in the corresponding part of the proof of Theorem 4.1.

□ □

4.6.2 The Price of Anarchy of Restricted Average-Oriented Games

We proceed to bound the **PoA** of restricted symmetric average-oriented games. Due to opinion restriction to $[0, 1]$, the average opinion at a Nash Equilibrium may be far from $\mathsf{avg}(s)$. Therefore, we cannot rely on Proposition 4.6 anymore. Moreover, the **PoA** of restricted games increases fast with α (e.g., if $s = (0, \ldots, 0, 1/n)$, $w_{ij} = 0$ for all $i \neq j$, and $\alpha = n^2$, $\mathsf{PoA} = \Omega(n)$). Therefore, we here restrict our attention to the case where $\alpha = w = 1$ and show that the **PoA** of restricted symmetric average-oriented games remains constant. An interesting intermediate result of our analysis is that if all agents only value the distance of their opinion to their belief and to the average, i.e., if $w_{ij} = 0$ for all $i \neq j$, the **PoA** of such games is at most $1 + 1/n^2$.

As in the proofs of Lemma 4.2 and Theorem 4.2, we use a generalized local smoothness argument. In this case, however, the function $\sum_{i=1}^{n}(\mathsf{avg}(y) - s_i)^2$

is not (λ, μ)-locally smooth and $\mathbf{avg}(y^*)$ at the equilibrium y^* may be far from $\mathbf{avg}(s)$. Hence, to bound the **PoA**, we need to advance substantially beyond the local smoothness argument of [15, Sec. 3.1]. The rest of this section is devoted to the proof of the following:

Theorem 4.4. *Let $\mathcal{G}$ be any symmetric average-oriented opinion formation game with $w = \alpha = 1$, $n \geq 2$ agents and opinions restricted to $[0, 1]$. Then,*

$$\mathbf{PoA}(\mathcal{G}) \leq 3 + \sqrt{2} + O(\frac{1}{n})$$

Proof. As in the proofs of Lemma 4.2 and Theorem 4.2, we seek to find appropriate parameters $\lambda > 0$ and $\mu \in (0, 1)$ such that for all $x, y \in [0, 1]^n$,

$$C(y) + (x - y)^T C'(y) \leq \lambda C(x) + \mu C(y)\,. \tag{4.12}$$

where $C(y) = \sum_{i=1}^n C_i(y)$ and $C'(y) = (\frac{dC_1(y)}{dy_1}, \cdots, \frac{dC_n(y)}{dy_n})$.

Next, we show that (4.12) indeed implies $\mathbf{PoA}(\mathcal{G}) \leq \lambda/(1-\mu)$. To this end, we show that at the equilibrium y^* of a restricted game, $(x - y^*)^T C'(y^*) \geq 0$. By definition $y^* \in [0, 1]^n$. In case where $y_i^* \in (0, 1)$, due to first-order optimality conditions, $\frac{dC_i(y^*)}{dy_i} = 0$ and $(x_i - y_i^*)\frac{dC_i(y)}{dy_i} = 0$. If $y_i^* = 0$ then $\frac{dC_i(y^*)}{dy_i} \geq 0$. Otherwise, agent i could decrease her cost by increasing y_i^*. Since $x_i \in [0, 1]$, $(x_i - y_i^*)\frac{dC_i(y^*)}{dy_i} \geq 0$. By a symmetric argument, if $y_i^* = 1$, $\frac{dC_i(y^*)}{dy_i} \leq 0$ and $(x_i - y_i^*)\frac{dC_i(y^*)}{dy_i} \geq 0$. Applying (4.12) for $y = y^*$ and $x = o^*$ (recall that the optimal solution $o^* \in [0, 1]^n$) yields

$$C(y^*) \leq C(y^*) + (o^* - y^*)^T C'(y^*) \leq \lambda C(o^*) + \mu C(y^*)\,.$$

Therefore, $\mathbf{PoA}(\mathcal{G}) = C(y^*)/C(o^*) \leq \lambda/(1 - \mu)$.

We proceed to establish (4.12). As in Section 4.5, in order to find appropriate values for λ and μ, we divide the individual cost of each agent i into two parts, writing $C_i(y) = F_i(y) + M_i(y)$, and analyze each part separately. We again have that:

$$\begin{aligned}
F(y) &= \sum_{i=1}^n F_i(y) = \sum_{i \in N} \sum_{j \neq i} w_{ij}(y_i - y_j)^2 \\
M(y) &= \sum_{i=1}^n M_i(y) = \sum_{i \in N} ((y_i - s_i)^2 + (\mathbf{avg}(y) - s_i)^2) \\
&= (y - s)^T (y - s) + (\mathbf{avg}(y) - s)^T (\mathbf{avg}(y) - s)\,.
\end{aligned}$$

We again denote

$$F'(y) = \left(\frac{dF_1(y)}{dy_1}, \cdots, \frac{dF_n(y)}{dy_n}\right)$$
$$M'(y) = \left(\frac{dM_1(y)}{dy_1}, \cdots, \frac{dM_n(y)}{dy_n}\right)$$

We also recall that $M'(y) = 2(y-s) + (2/n)(\mathbf{avg}(y) - s)$.

Proposition 4.4 provides an appropriate upper bound on the term $F(y) + (x-y)^T F'(y)$. So, we next focus on finding appropriate values of λ and μ so that we can bound from above the term $M(y) + (x-y)^T M'(y)$.

To this end, we first observe that:

$$M(y) + (x-y)^T M'(y) = M(y) + (s-y)^T M'(y) + (x-s)^T M'(y) \ .$$

We first bound $M(y) + (s-y)^T M'(y)$ from above using the following proposition. Intuitively, the proposition holds because the left-hand side of (4.13) is a strictly concave function of y.

Proposition 4.7. *For any $y, x, s \in [0,1]^n$,*

$$M(y) + (s-y)^T M'(y) \le (1 + \tfrac{1}{n^2}) M(s) \le (1 + \tfrac{1}{n^2}) M(x) \,. \tag{4.13}$$

Proof. Let $\mathbb{K}_n$ denote the $n \times n$ matrix with all its entries equal to $1/n$. Recall that $\mathbb{I}$ is the $n \times n$ identity matrix. Clearly, $\mathbb{K}_n y$ is the vector with all its coordinates equal to $\mathbf{avg}(y)$. Moreover, we observe that $\mathbb{K}_n \mathbb{K}_n = \mathbb{K}_n$. Using matrix notation, we obtain that:

$$\begin{aligned} M(y) + (s-y)^T M'(y) &= (\mathbb{K}_n y - s)^T (\mathbb{K}_n y - s) + (y-s)^T (y-s) \\ &\quad + 2(s-y)^T (y-s) + (2/n)(s-y)^T (\mathbb{K}_n y - s) \\ &= y^T ((1 - \tfrac{2}{n})\mathbb{K}_n - \mathbb{I}) y + 2\, y^T ((1 + \tfrac{1}{n})\mathbb{I} - (1 - \tfrac{1}{n})\mathbb{K}_n) s - \tfrac{2}{n} s^T s \ . \end{aligned}$$

We observe that the matrix $\mathbb{I} - (1 - \frac{2}{n})\mathbb{K}_n$ is strictly diagonally dominant, and thus positive definite. So, the matrix $(1 - \frac{2}{n})\mathbb{K}_n - \mathbb{I}$ is negative definite. Thus, $M(y) + (s-y)^T M'(y)$ is strictly concave in y and has a unique maximum in $\mathbf{R}$.

We next show that $M(y) + (s-y)^T M'(y)$ is maximized at $y^* = (1 + \frac{1}{n})s - \mathbf{avg}(s)/n$. To find the unique maximizer y^* of $M(y) + (s-y)^T M'(y)$, we apply first-order optimality conditions. The gradient of $M(y) + (s-y)^T M'(y)$ with respect to $y_1, \ldots, y_n$ is equal to

$$2((1 - \tfrac{2}{n})\mathbb{K}_n - \mathbb{I}) y + 2((1 + \tfrac{1}{n})\mathbb{I} - (1 - \tfrac{1}{n})\mathbb{K}_n) s \ .$$

So the unique maximizer y^* of $M(y) + (s-y)^T M'(y)$ satisfies

$$y_i^* = (1 + \tfrac{1}{n})s_i + (1 - \tfrac{2}{n})\mathsf{avg}(y^*) - (1 - \tfrac{1}{n})\mathsf{avg}(s) \ .$$

Summing up these equations for all $i \in N$, we obtain that

$$n\,\mathsf{avg}(y^*) = (n+1)\mathsf{avg}(s) + (n-2)\mathsf{avg}(y^*) - (n-1)\mathsf{avg}(s)\,,$$

which implies that $\mathsf{avg}(y^*) = \mathsf{avg}(s)$. Therefore, the maximizer y^* of $M(y) + (s-y)^T M'(y)$ has $y_i^* = (1+\frac{1}{n})s_i - \mathsf{avg}(s)/n$ (note in particular that y_i^* does not need to belong to $[0,1]$).

Using that $y^* = (1+\frac{1}{n})s - \mathsf{avg}(s)/n$ and $\mathsf{avg}(y^*) = \mathsf{avg}(s)$, we obtain:

$$\begin{aligned}
M(y^*) + (s-y^*)^T M'(y^*) &= -(y^*-s)^T(y^*-s) + (\mathsf{avg}(s)-s)^T(\mathsf{avg}(s)-s) \\
&\quad + (2/n)(y^*-s)^T(\mathsf{avg}(s)-s)^T \\
&= -(1/n^2)(\mathsf{avg}(s)-s)^T(\mathsf{avg}(s)-s) \\
&\quad + (\mathsf{avg}(s)-s)^T(\mathsf{avg}(s)-s) \\
&\quad + (2/n^2)(\mathsf{avg}(s)-s)^T(\mathsf{avg}(s)-s)^T \\
&= (1+\tfrac{1}{n^2})(\mathsf{avg}(s)-s)^T(\mathsf{avg}(s)-s) \ .
\end{aligned}$$

The proposition follows from the following observations: (i) for any $y \in [0,1]^n$, $M(y) + (s-y)^T M'(y) \le M(y^*) + (s-y^*)^T M'(y^*)$, since $y^* \in \mathbf{R}^n$ is the unique maximizer of the strictly concave function $M(y) + (s-y)^T M'(y)$; and (ii) for any $x \in [0,1]^n$,

$$\begin{aligned}
M(y^*) + (s-y^*)^T M'(y^*) &= (1+\tfrac{1}{n^2})(\mathsf{avg}(s)-s)^T(\mathsf{avg}(s)-s) \\
&= (1+\tfrac{1}{n^2})M(s) \le (1+\tfrac{1}{n^2})M(x)\,,
\end{aligned}$$

where the last inequality holds because s is a minimizer of $M(y)$. □

Remark 4.1. *If $w_{ij} = 0$ for all $i \neq j$, the cost of each agent i becomes $C_i(y) = (y_i - s_i)^2 + (\mathsf{avg}(y) - y_i)^2$. For this interesting class of restricted symmetric average-oriented games, Proposition 4.7 implies that the* PoA *is at most $1 + 1/n^2$.*

We proceed to show an upper bound on $(x-s)^T M'(y)$.

Proposition 4.8. *For any $y, x, s \in [0,1]^n$, and for any $\lambda_1, \lambda_2 > 0$ and $\mu_1, \mu_2 \in (0,1)$ such that $\lambda_1\mu_1 \ge 1$ and $\lambda_2\mu_2 \ge 1/n^2$,*

$$(x-s)^T M'(y) \le (\lambda_1 + \lambda_2)M(x) + \max\{\mu_1, \mu_2\}M(y)\,. \qquad (4.14)$$

Proof. We observe that

$$(x-s)^T M'(y) = 2(x-s)^T(y-s) + (2/n)(x-s)^T(\text{avg}(y)-s) \ .$$

Applying Proposition 4.3, with $\gamma = 1$, for each term $2(x_i - s_i)(y_i - s_i)$ of $2(x-s)^T(y-s)$, we obtain that for any $\lambda_1 > 0$ and $\mu_1 \in (0,1)$ with $\lambda_1\mu_1 \geq 1$,

$$2(x-s)^T(y-s) \leq \lambda_1(x-s)^T(x-s) + \mu_1(y-s)^T(y-s) \ .$$

Similarly, applying Proposition 4.3, with $\gamma = 1/n$, for each term $(2/n)(x_i - s_i)(\text{avg}(y) - s_i)$ of $(2/n)(x-s)^T(\text{avg}(y)-s)$, we obtain that for any $\lambda_2 > 0$ and $\mu_2 \in (0,1)$ with $\lambda_2\mu_2 \geq 1/n^2$,

$$(2/n)(x-s)^T(\text{avg}(y)-s) \leq \lambda_2(x-s)^T(x-s) + \mu_2(\text{avg}(y)-s)^T(\text{avg}(y)-s) \ .$$

Inequality (4.14) follows from summing up the two inequalities above and using that $M(x) \geq (x-s)^T(x-s)$ and that $M(y) = (y-s)^T(y-s) + (\text{avg}(y) - s)^T(\text{avg}(y)-s)$. □

Using Proposition 4.7 and Proposition 4.8, we obtain that for all $x, y \in [0,1]^n$, and for all $\lambda_1, \lambda_2 > 0$ and $\mu_1, \mu_2 \in (0,1)$ such that $\lambda_1\mu_1 \geq 1$ and $\lambda_2\mu_2 \geq 1/n^2$,

$$M(y) + (x-y)^T M'(y) \leq \left(1 + \frac{1}{n^2} + \lambda_1 + \lambda_2\right) M(x) + \max\{\mu_1, \mu_2\} M(y) \,. \tag{4.15}$$

Applying Proposition 4.4 with $\lambda = 1$ and $\mu = \sqrt{2} - 1$, and (4.15) with $\lambda_1 = \sqrt{2} + 1$, $\lambda_2 = 1/n$, $\mu_1 = \sqrt{2} - 1$ and $\mu_2 = 1/n$, and summing up the corresponding inequalities, we obtain that (4.12) holds with $\lambda = 2 + \sqrt{2} + \frac{n+1}{n^2}$ and $\mu = \sqrt{2} - 1$. Hence, we conclude that

$$\text{PoA} \leq (2 + \sqrt{2})^2/2 + (\sqrt{2} + 1)\tfrac{n+1}{n^2} \,.$$

□

Chapter 5

Network and Random Hegselmann Krause Model

In this Chapter, we investigate the convergence properties of the Network Hegselmann Krause model and the Random Hegselmann Krause model. Both of these models were introduced in our work [77] to capture different aspects of the well-known HK model. A brief introduction to Network HK model and Random HK model can be found in Sections 2.5.3 and 2.5.4 respectively.

5.1 Network Hegselmann Krause Model

In the Network HK model, we are given an undirected graph $G(V, E)$ where V stands for the agents and E the social relations among them. Each agent $i \in V$ initially holds an opinion $x_i(0) \in [0, 1]$. At each round, each agent averages her current opinion with the opinions of her *neighbors* that are ε-close to hers. The parameter $\varepsilon > 0$ measures the tolerance of the agents against different opinions.

Network Hegselmann Krause model

1: undirected graph $G = (V, E)$.
2: n agents.
3: $x_i(0) \in [0, 1]$, agent i's initial opinion.
4: At round $t \geq 1$, each agent i updates her opinion:

$$x_i(t) = \frac{\sum_{j \in N_i(t)} x_j(t-1) + x_i(t-1)}{|N_i(t)| + 1}$$

where $N_i(t) = \{j \in V : \ |x_i(t-1) - x_j(t-1)| \leq \epsilon \text{ and } (i, j) \in E\}$

Given the graph G, the initial opinions and the parameter ε, one can compute the opinions of the agents at any round t. Such a selection defines an opinion dynamics $x(t)$.

Definition 5.1. *An instance of the Network HK model is denoted by the triple* $(G, x(0), \varepsilon)$ *and* $x(t) \in [0, 1]^n$ *denotes the opinion vector at round* t.

In this chapter we shed light on the convergence properties of the Network HK model. Our results provide a positive answer to the following question.

Question 4. *Does the opinion vector* $x(t)$ *stabilizes to a stable state for any instance* $(G, x(0), \varepsilon)$*?*

Every instance of the Network HK model admits an infinite number of stable opinion vectors. Assume that at some point in time the opinions have the following form: Each agent either has opinion x_1 or opinion x_2 where $|x_1 - x_2| > \varepsilon$. Clearly the system will remain in this state forever. The agents with opinion x_1 average their opinion with the opinions of their x_1-neighbors, while the same happens with those with opinion x_2. Theorem 5.1 states that such a stable state is always reached by the system.

Theorem 5.1. *For any instance* $(G, x(0), \varepsilon)$ *of the Network HK model, the opinion vector* $x(t)$ *reaches a stable state.*

Section 5.2 is dedicated to the proof of Theorem 5.1.

5.2 Convergence of Network Hegselmann Krause Model

A crucial step for proving Theorem 5.1 is describing the opinion dynamics as the following matrix product.

Corollary 5.1. *For any instance* $(G, x(0), \varepsilon)$ *of the Network HK model, the opinion vector* $x(t)$ *can be written in the following matrix form:*

$$x(t) = A^t x(t-1) = A^t \cdots A^1 x(0)$$

$$\textit{where } A^t_{ij} = \begin{cases} \frac{1}{|N_i(t)|+1} & \textit{if } j = i \\ \frac{1}{|N_i(t)|+1} & \textit{if } j \in N_i(t) \\ 0 & \textit{otherwise} \end{cases}$$

Each matrix A^t is stochastic (has positive elements and the sum of each row equals 1), has positive diagonal elements and has the following symmetric property, if $A^t_{ij} > 0$ then $A^t_{ji} > 0$. As we will see latter, the third property is of great importance for establishing the convergence properties of the Network HK model.

Each matrix A^t can also be perfectly represented by an undirected graph that is an induced subgraph of G and consists of the *activated edges* of E at round t. By the term *activated edges* we mean the pairs $(i, j) \in E$ such that $|x_i(t) - x_j(t)| \leq \varepsilon$. Probably with some abuse of terminology, throughout this section we refer to A^t either as a matrix or graph. This «dual» consideration extremely simplifies things and provide us with a lot of intuition on why Network HK model always reaches a stable state. This intuition is presented after Definition 5.2.

Definition 5.2. *Let the partition $V = (S, V/S)$ then $\delta^t(S, V/S)$ denotes the edges of A^t between S and V/S or equivalently the set of pairs (i, j) where $i \in S, j \in V/S$ such that $A^t_{ij} > 0$.*

Assume that graph G has two connected components G_1, G_2. Then the overall system breaks into two independent subsystems since the agents of G_1 are never influenced by the agents of G_2 and vice versa. As a result, without loss of generality we can assume that G is connected. Describing the system as a graph-matrix sequence $A^1, \dots, A^t, \dots$ permits us to apply a similar observation on the time domain. Assume that there exists a round t_0 such that for all $t \geq t_0$, $\delta^t(S, V/S) = \varnothing$. The latter means that after t_0 there is no interaction between any agents in S and V/S and thus the system *breaks* into independent subsystems. Since at most $|V| - 1$ breaks can occur, after a finite round no break happens. Thus without loss of generality, we can assume that a *break* never takes place. Definition 5.3 and Corollary 5.2 establish the above intuition in a formal way.

Definition 5.3. *A set of agents $S \subseteq V$ is* ***weakly connected*** *if and only if for any non-empty $S' \subset S$ and any $t_0 \in \mathbb{N}$, there is a round $t \geq t_0$ so that A^t includes at least one edge connecting an agent in S' to some agent in $S \backslash S'$.*

Definition 5.3 is the negation of the property that a break takes place. An instance of Network HK model in which V is weakly connected, is presented in Example 5.1.

Example 5.1. *Let an instance of the Network HK model where G is connected and for all $i, j \in V$, $|x_i(0) - x_j(0)| \leq \varepsilon$. The respective graph-matrix sequence is $G, \dots G, \dots$ meaning that V is weakly connected.*

Corollary 5.2. *Let $(G, x(0), \varepsilon)$ an instance of the Network HK Model. Then there exists a round t^* and a partition of $V = (V_1, V_2, \ldots, V_k)$ such that*

- *each V_ℓ is weakly connected.*
- *for all $t \geq t^*$, $\delta^t(V_\ell, V \backslash V_\ell) = \varnothing$.*

Proof. Corollary 5.2 directly follows by induction on the number of nodes and by the definition of weak connectivity. □

We can now vividly explain the significancy of the notion of weak connectivity for proving that the Network HK model always reaches a stable state. The reasoning procceeds as follows: If in the given instance $(G, x(0), \varepsilon)$, V is not weakly connected then at some finite round the system breaks into independent subsystems that are weakly connected (Corollary 5.2). Thus without loss of generality we can assume that V is weakly connected. In case V is weakly connected we argue that the influences among the agents are so strong, that finally all agents adopt the same opinion! This is formally stated in Theorem 5.2 that is the main result of the section.

Theorem 5.2. *Let $(G, x(0), \varepsilon)$ an instance of Network HK model such that V is weakly connected. Then there exits $t_0 \in \mathbb{N}$ such that*

$$x_i(t_0) = x_j(t_0), \text{ for all } i, j \in V$$

As mentioned above, Theorem 5.2 is the major result of the section. Theorem 5.1 follows by direct application of Lemma 5.2 and Theorem 5.2. For the sake of completeness we present the proof and then we dedicate the rest of the section to prove Theorem 5.2.

Theorem 5.1. *For any instance $(G, x(0), \varepsilon)$ of the Network HK model, the opinion vector $x(t)$ reaches a stable state.*

Proof. By Lemma 5.2 there exists t^* and a partition of $V = (V_1, V_2, \ldots, V_k)$ such that

- each V_ℓ is weakly connected.
- for all $t \geq t^*$, $\delta^t(V_\ell, V \backslash V_\ell) = \varnothing$.

Due to the second condition, the opinions of the agents in each V_ℓ after round t^*, equal the opinions of the agents of the instance $(G_{V_\ell}, x_{V_\ell}(t^*), \varepsilon)$. Due to the first condition in each instance $(G_{V_\ell}, x_{V_\ell}(t^*), \varepsilon)$ the set V_ℓ is weakly connected and thus Theorem 5.2 applies. □

Theorem 5.2 states that if V is weakly connected then at same point in time all the agents will adopt the same opinion. In a sense Theorem 5.2 states that the rank of the matrix-product $A^t \cdots A^1$ converges to 1 at t grows. This intuition is formally stated in Theorem 5.4 which implies Theorem 5.2. Before presenting Theorem 5.4 we present the notion of the *coefficient of ergodicity* [132], which is a very useful tool for studying products of stochastic matrices.

Definition 5.4. *Let A be a stochastic matrix then the coefficient of ergodicity of matrix A,*

$$\tau(A) = \frac{1}{2} \cdot \max_{i,j} \sum_{k=1}^{n} |A_{ik} - A_{jk}|$$

and has the following properties:

- $\tau(A \cdot B) \leq \tau(A) \cdot \tau(B)$
- *if A has positive elements then $\tau(A) < 1$*
- $\tau(A) = 0$ *if and only if rank(A)=1*

We are now ready to state Theorem 5.4, which implies the Theorem 5.2.

Theorem 5.4. *Let the graph-matrix sequence $A^1, \ldots, A^t, \ldots$ of an instance $(G, x(0), \varepsilon)$ of the Network HK model in which V is weakly connected. Then,*

$$\lim_{t\to\infty} \tau(A^t \cdots A^1) = 0$$

Before exhibiting the proof of Theorem 5.4, we present the proof of Theorem 5.2

Theorem 5.2. *Let $(G, x(0), \varepsilon)$ an instance of Network HK model such that V is weakly connected. Then there exits $t_0 \in \mathbb{N}$ such that*

$$x_i(t_0) = x_j(t_0), \text{ for all } i, j \in V$$

Proof. Since V is weakly connected by Theorem 5.4, $\lim_{t\to\infty} \tau(A^t \cdots A^1) = 0$. As a result, there exists a round t_0 such that the coefficient of ergodicity of stochastic matrix $C = A^{t_0} \cdots A^1$ is $\tau(C) \leq \varepsilon/2$. Since $x(t_0) = Cx(0)$ we have that for all i and j,

$$\begin{aligned} |x_i(t_0) - x_j(t_0)| &= |(C_i - C_j)x(0)| \\ &\leq \|C_i - C_j\|_1 \\ &\leq 2\tau(C) \leq \varepsilon \end{aligned}$$

where C_i is the i-th row of matrix C. Since at t_0 all opinions are within distance ε we have that A^{t_0} equals G (enhanced with self loops). Moreover $\tau(A^t \cdots A^{t_0} \cdots A^1) \leq \tau(A^{t_0} \cdots A^1) \leq \varepsilon/2$, meaning that for all $t \geq t_0$,

$$A^t = A^{t_0}$$

Hence, after round t_0, we have essentially an instance of DeGroot's model on the undirected connected network G (enhanced with self-loops), which fulfills the conditions for convergence. Moreover, all agents converge to a single opinion [92]. □

We complete the section with the proof of Theorem 5.4. The proof follows directly from the submultiplicative propery of the coefficient of ergodicity and Lemma 5.1.

Lemma 5.1. *Let the graph-matrix sequence $A^1, \ldots, A^t, \ldots$ of an instance $(G, x(0), \varepsilon)$ of the Network HK model in which V is weakly connected. Then, for any $t_0 \in \mathbb{N}$ there exists $\ell(t_0) \in \mathbb{N}$ such that*

$$\tau(A^{\ell(t_0)} \cdots A^{t_0}) \leq 1 - (1/n)^{n^2}$$

Proof. We will use the fact that V is weakly connected to prove that for any t_0 there exists a round $\ell(t_0)$, such that the matrix product $C^{\ell(t_0)}$ has all of its elements positive ($C^t = A^t \cdots A^{t_0}$). Then, by the properties of coefficient of ergodicity $\tau(C^{\ell(t_0)}) < 1$.

Notice that the element C^t_{ij} is positive if and only if there is a (time-respecting) walk $(i = u_0, u_1, \ldots, u_{t-t_0} = j)$ from node i to node j such that the edge $\{u_k, u_{k+1}\}$ exists in A^{t_0+k}. Recall that any matrix A^t has positive diagonal elements or equivalently every node in the graph A^t has a self loop. Thus, if $C^{t-1}_{ij} > 0$ then $C^t_{ij} > 0$, since the time respecting walk from i to j can use the self loop of node j. The latter implies that $\text{Pos}_i(t-1) \subseteq \text{Pos}_i(t)$, where $\text{Pos}_i(t)$ denotes the positive elements at the i-th row of C^t (equivalently the nodes reachable from i in $t - t_0 + 1$ steps). Since V is weakly connected, there exists a time step $t' > t$ such that $A^{t'}$ contains an edge $\{j, m\}$ traversing the cut $(\text{Pos}_i(t), V \backslash \text{Pos}_i(t))$. Provided that $j \in \text{Pos}_i(t) \subseteq \text{Pos}_i(t')$ and $\{j, m\} \in E_{A_{t'}}$ shows that $m \in \text{Pos}(t')$. Thus, $|\text{Pos}_i(t)| + 1 \leq |\text{Pos}_i(t')|$ and repeating the same argument for all the rows of C^t proves our claim.

Up next, we prove that $\tau(C^{\ell(t_0)}) \leq 1 - (1/n)^{n^2}$. Observe that in the previous proof we have implicitly mentioned two types of matrices participating in the product $A^{\ell(t_0)} \cdots A^{t_0}$. There are the ones that augment the total positive elements in the overall product and those who preserve the positive elements through the use of self loops. We call these two types of matrices *expanding* and *non-expanding* respectively. More precisely, A^t is *expanding* if and only if $\text{Pos}(C^{t-1}) \subsetneq \text{Pos}(C^t$, where $\text{Pos}(M)$ is the set of positive elements of matrix M. Recall, from the previous paragraph, that $C^t = A^t \cdot C^{t-1}$ and $\text{Pos}(C^{t-1}) \subseteq \text{Pos}(C^{t-1})$. At first, we prove that in case A^t is an *non-expanding* ($\text{Pos}(C^t) = \text{Pos}(C^{t-1})$) the minimum positive element of C^t is greater than the minimum positive element of C^{t-1}.

Let δ be the minimum positive element of C^{t-1}. Since $\text{Pos}(A^t \cdot C^{t-1}) = \text{Pos}(C^{t-1})$, we just need to show that if $C_{ij}^{t-1} > 0$ then $(A^t \cdot C^{t-1})_{ij} \geq \delta$. Suppose that $C_{ij}^{t-1} > 0$ then

$$(A^t C^{t-1})_{ij} = \sum_{l=1}^{n} A_{il}^t C_{lj}^{t-1} = \sum_{l:C_{lj}^{t-1}>0} A_{il}^t C_{lj}^{t-1}$$

We will prove that $\sum_{l:C_{lj}^{t-1}>0} A_{il}^t = 1$ and this directly implies our claim. Let us assume that $\sum_{l:C_{lj}^{t-1}>0} A_{il}^t < 1$. This means that there exists k s.t. $A_{ik}^t > 0$ and $C_{kj}^{t-1} = 0$. Since $\text{Pos}(A^t \cdot C^{t-1}) = \text{Pos}(C^{t-1})$ and $C_{kj}^{t-1} = 0$ then $(A^t C^{t-1})_{kj} = 0$. Observe that $(A^t C^{t-1})_{kj} \geq A_{ki}^t \cdot C_{ij}^{t-1} \implies A_{ki}^t \cdot C_{ij}^{t-1} = 0 \xRightarrow{C_{ij}^{t-1}>0} A_{ki}^t = 0$. Finally we get $A_{ki}^t = 0$ and $A_{ik}^t > 0$. This can't be true, because $A_{ik}^t > 0$ implies that $\{i,k\} \in E$ and $|x_i(t) - x_k(t)| \leq \epsilon$ meaning that $A_{ki}^t > 0$.

The matrix product $C^{\ell(t_0)} = A^{\ell(t_0)} \cdots A^{t_0}$ contains at most n^2 expanding steps (the number of positive elements in $C^{\ell(t_0)}$ is n^2). As a result, the minimum positive element of A^{t_0} decreases only n^2 times. Since the minimum positive element of any martix A^t is $1/n$, the minimum positive element of $C^{\ell(t_0)}$ is greater than $(1/n)^{n^2}$. Combining this with the fact that all elements of $C^{\ell(t_0)}$ are positive, we get that $\tau(C^{\ell(t_0)}) \leq 1 - (1/n)^{n^2}$. □

Remark 5.1. *We remark that this proof can be generalized to prove convergence of the d-dimensional Network HK model. In this case each agent i maintains a d-dimensional opinion vector $x_i(t) \in [0,1]^d$ and the update rule is defined respectively by the d-dimensional HK model [14] and a social network G. The proof is essentially identical, with the only difference that we need to prove the existence of a time step t_0 such that $\tau(C) \leq \varepsilon/(2\sqrt{d})$, where $C = A^{t_0} \cdots A^0$. But, we have already proven that $\lim_{t\to\infty} \tau(A^t \cdots A^0) = 0$.*

5.3 Random Hegselmann Krause Model

In the Random HK model, each agent i initially holds an opinion $x_i(0) \in [0,1]$. At each round t, each agent picks k other agents (including herself) uniformly at random with replacement. Then she averages her current opinion with the opinions of those that are ε-close to hers.

Random Hegselmann Krause model

n agents.
$x_i(0) \in [0,1]$, agent i's initial opinion.
At round $t \geq 1$, each agent i:
  selects k agents uniformly at random with replacement, $R_i(t) \subseteq [n]$
  updates her opinion,

$$x_i(t) = \frac{\sum_{j \in N_i(t)} x_j(t-1) + x_i(t-1)}{|N_i(t)| + 1}$$

where $N_i(t) = \{j : \ |x_i(t-1) - x_j(t-1)| \leq \varepsilon \text{ and } j \in R_i(t)\}$

As already discussed, Random HK model is a straightforward variant of the original HK model, in which each agents meets just a small subset of the other agents at each round. In this chapter we prove that the convergence properties of the HK model are preserved even in this random and limited information exchange setting.

Since Random HK model is a stochastic opinion dynamics, the selection of the initial opinion, the sampling size of the agents and the parameter ε defines a probability distribution over the opinion vector $x(t)$.

Definition 5.5. *An instance of Random HK model is denoted by* $(x(0), k, \varepsilon)$ *and* $x(t)$ *is the produced opinion vector at round* t*.*

The convergence properties of the Random HK model are depicted in Theorem 5.6. Before presenting it, we introduce some neccessary notions that are neccessary both in stating Theorem 5.6 and in its subsequent analysis.

Definition 5.6. *Let* S_1, S_2 *two disjoint sets of agents, we denote their distance at round* t *as*

$$d^t(S_1, S_2) = \min_{i \in S_1, j \in S_2} |x_i(t) - x_j(t)|$$

Definition 5.7. *A set of agents* S *is* ε*-connected at round* t*, if and only if for any non-empty set* $S' \subset S$*,*

$$d^t(S', S \setminus S') \leq \varepsilon$$

Definition 5.8. *The diameter at round* t*, denoted* $Diam(t)$*, is the maximum distance* $|x_i(t) - x_j(t)|$ *over all pairs of agents* i, j *in the same* ε*-connected component at round* t*.*

Theorem 5.6. *Let $(x(0), k, \varepsilon)$ be any instance of the Random HK model. For any $\gamma, \delta > 0$ there is a round t^* such that for all $t \geq t^*$:*

$$\mathbf{P}\left[Diam(x(t)) \leq \gamma\right] \geq 1 - \delta$$

Theorem 5.6 states that agents form opinion clusters with inter-cluster distance at least ε. More precisely, if we «look» the system after a large number of rounds: the distance of the opinions of any two agents $|x_i(t) - x_j(t)|$ will be either less than γ (which can be made arbitrarily small) or greater than ε. Notice that if $|x_i(t) - x_j(t)| \in (\gamma, \varepsilon]$ then agents i, j must be in the same ε-connected component, meaning that the diameter $Diam(x(t)) > \gamma$, which contradicts Theorem 5.2. Moreover these clusters remain the same. Two different ε-connected components can never be merged since the distance of any two agents from two different components is at least ε. At the same time if $\gamma \leq \epsilon$, an ε-connected component cannot *break* since the maximum distance of the opinions is at most γ and this cannot increase no matter the random meetings.

5.4 Convergence of Random Hegselmann Krause Model

The goal of this section is to prove Theorem 5.6. Although the strategy proof resembles that of Section 5.2, there are some major differences that are explained up next. As in Section 5.2, the basic step is to describe the opinion dynamics as a product of stochastic matrices.

Corollary 5.3. *For any instance $(x(0), k, \varepsilon)$ of the Random HK model, the opinion vector $x(t)$ can be written in the following matrix form:*

$$x(t) = A^t x(t-1) = A^t \cdots A^1 x(0)$$

$$\text{where } A^t_{ij} = \begin{cases} \frac{1}{|N_i(t)|+1} & \text{if } j = i \\ \frac{1}{|N_i(t)|+1} & \text{if } j \in N_i(t) \\ 0 & \text{otherwise} \end{cases}$$

Each matrix A^t is stochastic and has positive diagonal elements. Moreover these matrices are random variables since they depend of the realization of the random meetings of the agents. As in Section 5.2, each matrix A^t can also be represented as graph in which an edge (i, j) exists if and only if $A^t_{ij} > 0$.

The major difference between the Random HK model and the Network HK model is that the resulting graph of A^t can be directed. For example consider the case where $|x_i(t) - x_j(t)| \leq \varepsilon$ and i picks j, but j does not pick i. This asymmetry in the influence does not seem of great importance, but in fact HK systems with such asymmetric influence are far from being well understood [37]. From a technical point of view, Lemma 5.1 does not apply since it requires that if $A^t_{ij} > 0$ then $A^t_{ji} > 0$, the influence among the agents is «symmetric».

In order to study the convergence properties of the Random HK model, we first seek for conditions under which a subdivision of the system occurs. This is captured through the notion of ε-connectivity introduced in Definition 5.7. Consider two different ε-connected components $S, V \backslash S$ at round t_0. For all $i \in S$ and $j \in V \backslash S$, $|x_i(t_0) - x_j(t_0)| > \varepsilon$. It is not hard to see that no matter the random meetings of the agents, $|x_i(t) - x_j(t)| > \varepsilon$ for all rounds $t \geq t_0$. This means that after round t_0, the agents in S are not influenced by the agents in $V \backslash S$ and thus the system is separated into two independent subsystems.

Definition 5.9. *A set of agents S **breaks at round** t if and only if S is ε-connected at round $t-1$ and is not ε-connected at round t.*

As already discussed, once S' and $S \backslash S'$ break, they behave as independent instances of the Random HK model. Notice that at most $n-1$ breaks can occur, meaning that the event of a break, automatically reduces the number of future breaks. This provides some intuition on how the system performs. Assume that the system runs for a long period during which a small number of breaks take place. At the end of the period, the opinions of the nodes in each ε-connected component would be similar, since there would be a great deal of interaction between them, preventing the event of future breaks. On the other hand, a large number of breaks (during this time period) reduces the number of future breaks and consequently the first case applies. The following definitions and lemmas formalize the above intuition.

Definition 5.10. *We denote as Γ_ℓ the set of all instances (y, k, ε) of Random HK model, in which for all rounds $t \geq 0$,*

$$\mathbf{P}\left[\text{at most } \ell \text{ breaks occur in } \{0, t\} \,|\, x(0) = y\right] = 1$$

The set Γ_ℓ consists of all vectors $y \in [0, 1]^n$ such that if the initial opinions are y, then no matter the random choices of the agents, at most ℓ breaks occur.

Example 5.2. *Consider the instance* $(x(0), k, \varepsilon)$ *such that for all* i, j,

$$|x_i(0) - x_j(0)| \leq \varepsilon$$

In such an instance, $\max_i x_i(t) - \min_i x_i(t) \leq \varepsilon$ *for all rounds* t*, no matter the random meetings. As a result, no break ever occurs and thus* $(x(0), k, \varepsilon) \in \Gamma_0$.

In Lemma 5.2 we prove that if no break ever takes place, then there would be enough influence among the agents that leads them in adopting similar opinions, which is the first case of the above presented high level intuition. We show that if an instance $(x(0), k, \varepsilon) \in \Gamma_0$ then the agents adopt similar opinions with high probability.

Lemma 5.2. *Let an instance of the Random HK model,* $(x(0), k, \varepsilon) \in \Gamma_0$. *For any* $\gamma, \delta > 0$*, there is a round* t_0 *such that for all* $t \geq t_0$:

$$\mathbf{P}\left[Diam(t) \leq \gamma\right] \geq 1 - \delta$$

Proof. Without loss of generality, we assume that there exists a single ε-connected component since otherwise we can amplify the probability over the, at most n, ε-connected components.

We note that if $|x_i(t) - x_j(t)| \leq \varepsilon$, then the probability that agent j is at i's sample set at round t, is $p = 1 - (1 - 1/n)^k$. For any round ℓ, we denote $C^t = A^{t+\ell} \cdots A^\ell$ and $D^\ell = A^{\ell-1} \cdots A^1$. We claim that there is a fixed $\eta > 0$ such that for any possible matrix D^ℓ,

$$\mathbf{E}\left[\tau(C^{2n^2/p})|D^\ell\right] \leq 1 - \eta/2$$

Let $\text{Pos}_i(t)$ denotes the random set of positive elements of the i-th row of the matrix C^t. Assume that $\sum_{i=1}^n |\text{Pos}_i(t)| < n^2$ then there exists i such as

$$|\text{Pos}_i(t)| \leq n - 1$$

Since our instance belongs in Γ_0 then no break ever occurs and thus

$$d^t(\text{Pos}_i(t), V \backslash \text{Pos}_i(t)) \leq \varepsilon$$

This implies that there exists $u \in \text{Pos}_i(t)$, $j \in V \backslash \text{Pos}_i(t)$ such that $|x_u(t) - x_j(t)| \leq \varepsilon$. Since u chooses j with probabilty at least p the expected number of rounds, before all the elements of C^t become positive is at most n^2/p. By Markov Inequality,

$$\mathbf{P}\left[\tau(C^{2n^2/p}) < 1|\, D^\ell\right] \leq 1/2$$

where $\tau(\cdot)$ is the coefficient of ergodicity (see Definition 5.4 of Section 5.2). Since $C^{2n^2/p}$ is the product of $2n^2/p$ matrices, there exists a fixed $\eta > 0$ such that if $\tau(C^\ell) < 1$ then

$$\tau(C^\ell) \leq 1 - \eta$$

Thus, we get that for any fixed value of D^ℓ,

$$\mathbf{E}\left[\tau(C^\ell)|D^\ell\right] \leq 1 - \eta/2$$

We can now obtain a matrix $C = A^{t_0} \cdots A^1$ such that $\tau(C) \leq \gamma/2$ with probability at least $1 - \delta$, by taking an appropriately large number of rounds. □

Lemma 5.2 provides us with the an efficient primitive for establishing Theorem 5.6. If the system starts at a Γ_0 state, then Theorem 5.6 follows by a direct application of Lemma 5.2. In Lemma 5.3, we prove that no matter the initial opinion vector the system «falls» in a Γ_0-state with probability 1. Interestingly its proof uses Lemma 5.2.

Lemma 5.3. *Let* $(x(0), k, \varepsilon)$ *be any instance of the* Random HK model. *For any* $\delta^* > 0$ *there is a round* t^* *such that*

$$\mathbf{P}\left[x(t^*) \in \Gamma_0\right] \geq 1 - \delta^*$$

Proof. Let t_0 be the number of rounds in Lemma 5.2 for $\gamma = \varepsilon$. By definition if $x(0) \in \Gamma_0$ then

$$\mathbf{P}\left[Diam(x(t_0)) \leq \varepsilon\right] \geq 1 - \delta$$

We first present the high level idea of the proof. Assume that the systems does not start at Γ_0, but a break cannot occur in the first t_0 steps. This means that *breaks* start to appear after t_0 round. The basic observation is that if this is true then Lemma 5.2 applies and by definition of t_0, $Diam(x(t_0)) \leq \varepsilon$ with probability at least $1 - \delta$. This implies that with probability at least $1 - \delta$ the system falls in a Γ_0-state. If this is not the case, that is a break can occur in the first t_0 rounds, implies that there is at least one sequence of length $k \cdot n \cdot t_0$ describing the random meetings of the agents that leads the system to a *break*. Since such a sequence can be seleceted by the agents with probability at least $p = 1/n^{k \cdot n \cdot t_0}$, the system goes from a Γ_ℓ state to a $\Gamma_{\ell-1}$ with probability at least p, meaning that we will end up to a Γ_0-state. A rigourous version of this informal proof is presented up next.

We claim that $\mathbf{P}\left[x(t_0) \in \Gamma_{\ell-1} | x(0) \in \Gamma_\ell\right] \geq (1 - \delta)p$, where $p = 1/n^{k \cdot n \cdot t_0}$. Notice that p is the probability that a specific random sequence of length t_0 is selected. In order to prove our claim, we have to examine two mutually exclusive cases:

$\mathbf{P}\left[a \textit{ break occurs in } \{0, t_0\}\right] = 0:$ Since no break occurs in $\{0, t_0\}$ for all random choices of the agents, Lemma 5.2 can be applied. By definition of t_0, we have that $\mathbf{P}\left[Diam(x(t_0)) \leq \epsilon\right] \geq 1 - \delta$.

$$\begin{aligned}\mathbf{P}\left[x(t_0) \in \Gamma_{\ell-1} | x(0) \in \Gamma_\ell\right] &\geq \mathbf{P}\left[Diam(x(t_0)) \leq \epsilon | x(0) \in \Gamma_l\right] \\ &\geq 1 - \delta \geq (1-\delta)p\end{aligned}$$

$\mathbf{P}\left[a \textit{ break occurs in } \{0, t_0\}\right] > 0:$ The latter ensures the existence of a random sequence of length t_0 such that a break takes place. This sequence is selected with probability at least p. Implying the existence of an opinion vector $y \in \Gamma_{l-1}$ and $\mathbf{P}\left[x(t_0) = y\right] \geq p$. Hence,

$$\mathbf{P}\left[x(t_0) \in \Gamma_{\ell-1} | x(0) \in \Gamma_\ell\right] \geq p \geq (1-\delta)p$$

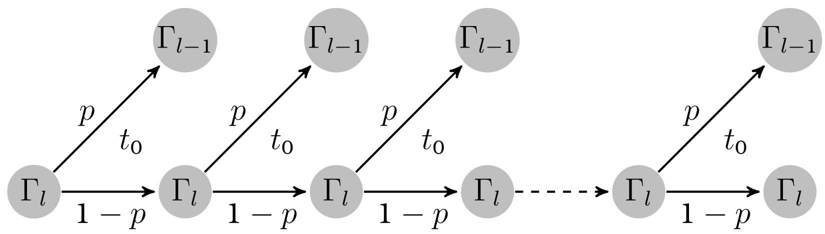
Until now, we have shown the existence of parameters t_0, δ, p that depend only the instance $(x(0), k, \varepsilon)$ and $\mathbf{P}\left[x(t_0) \in \Gamma_{l-1} | x(0) \in \Gamma_\ell\right] \geq (1-\delta)p$. Because our process is memoryless $\mathbf{P}\left[x(t+t_0) \in \Gamma_{\ell-1} | x(t) \in \Gamma_\ell\right] \geq (1-\delta)p$, holds for all $t \in \mathbb{N}$. Since at most $n-1$ breaks can occur, we conclude that $x(0) \in \Gamma_{n-1}$ and the proof follows directly from random walks on a chain graph, see Figure 5.1. □

Theorem 5.6. *Let $(x(0), k, \varepsilon)$ be any instance of the Random HK model. For any $\gamma, \delta > 0$ there is a round t^* such that for all $t \geq t^*$:*

$$\mathbf{P}\left[Diam(x(t)) \leq \gamma\right] \geq 1 - \delta$$

Proof. By Lemma 5.3, for any δ' there exists t' such that $\mathbf{P}\left[x(t') \in \Gamma_0\right] \geq 1 - \delta'$. Then Theorem 5.6 follows by direct application of Lemma 5.1. □

We conclude the section by summarize the proof of convergence of the Random HK model. At first, Lemma 5.2 ensures that there exists $t^* \in \mathbb{N}$ such that $x(t^*) \in \Gamma_0$ and then Lemma 5.3 ensures convergence to a single opinion in each ε-connected component.


Γ_{l-1}
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$1-p$

Figure 5.1

Chapter 6

Reallocating Facilities on the Line

In this chapter we present a polynomial time algorithm for the *K-Facility Reallocation Problem* that was introduced in [58]. The presented results are part of our work in [76]. A brief introduction to this problem can also be found in Section 2.5.5.

6.1 Problem Definition and Preliminaries

Definition 6.1 (K-Facility Reallocation Problem)**.** *We are given a tuple (x^0, C) as input. The K dimensional vector $x^0 = (x_1^0, \dots, x_K^0)$ describes the initial positions of the facilities. The positions of the agents over time are described by $C = (C_1, \dots, C_T)$. The position of agent i at stage t is α_i^t and $C_t = (\alpha_1^t, \dots, \alpha_n^t)$ describes the positions of the agents at stage t.*

Definition 6.2. *A solution of K-Facility Reallocation Problem is a sequence $x = (x^1, \dots, x^T)$. Each $x^t = (x_1^t, \dots, x_K^t)$ is a K dimensional vector that gives the positions of the facilities at stage t and x_k^t is the position of facility k at stage t. The cost of the solution x is*

$$Cost(x) = \sum_{t=1}^{T} \left[\sum_{k=1}^{K} |x_k^t - x_k^{t-1}| + \sum_{i=1}^{n} \min_{1 \le k \le K} |\alpha_i^t - x_k^t| \right]$$

Given an instance (x^0, C) of the problem, the goal is to find a solution x that minimizes the $Cost(x)$. The term $\sum_{t=1}^{T} \sum_{k=1}^{K} |x_k^t - x_k^{t-1}|$ describes the cost for moving the facilities from place to place and we refer to it as *moving cost*, while the term $\sum_{t=1}^{T} \sum_{i=1}^{n} \min_{1 \le k \le K} |\alpha_i^t - x_k^t|$ describes the connection cost of the agents and we refer to it as *connection cost*.

6.2 Solving the K-Facility Reallocation Problem in Polynomial Time

Our approach is a typical LP based algorithm that consists of two basic steps.

- **Step 1:** Expressing the *K-Facility Reallocation Problem* as an Integer Linear Program.
- **Step 2:** Solving *fractionally* the Integer Linear Program and *rounding* the fractional solution to an integral one.

6.2.1 Formulating the Integer Linear Program

A first difficulty in expressing the *K-Facility Reallocation Problem* as an Integer Linear Program is that the positions on the real line are infinite. We remove this obstacle with help of the following lemma proved in [58].

Lemma 6.1. *Let (x_0, C) an instance of the K-facility reallocation problem. There exists an optimal solution x^* such that for all stages $t \in \{1, T\}$ and $k \in \{1, K\}$,*

$$x_k^{*t} \in C_1 \cup \ldots \cup C_T \cup x^0$$

According to Lemma 6.1, there exists an optimal solution that locates the facilities only at positions where either an agent has appeared or a facility was initially lying (see Figure 6.1). Lemma 6.1 provides an exhaustive search algorithm for the problem and is also the basis for the *Dynamic Programming* approach in [58]. We use Lemma 6.1 to formulate our Integer Linear Program.

The set of positions $Pos = C_1 \cup \ldots \cup C_T \cup x^0$ can be represented equivalently by a path $P = (V, E)$. In this path, the j-th node corresponds to the j-th

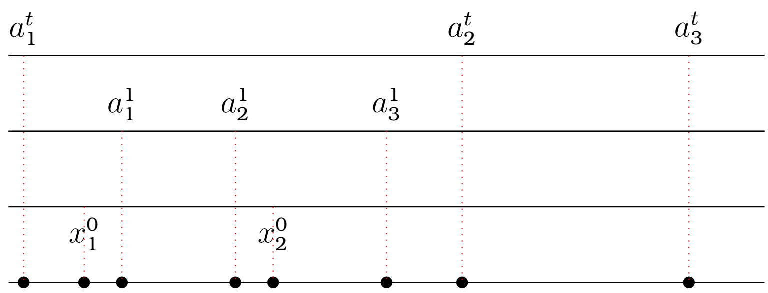


Figure 6.1: According to Lemma 6.1, there exists an optimal solution that opens facilities only to positions in which a facility was initially lying or a request has appeared at some point in time.

$$\min \textstyle\sum_{t=1}^{T} \left[\sum_{i\in C} \sum_{j\in V} d(\mathrm{Loc}(i,t),j)x_{ij}^t + \sum_{k\in F} S_k^t \right]$$

$$\sum_{j\in V} x_{ij}^t = 1 \qquad \forall i \in C, t \in \{1,T\}$$

$$x_{ij}^t \leq c_j^t \qquad \forall i \in C, j \in V, t \in \{1,T\}$$

$$c_j^t = \sum_{k\in F} f_{kj}^t \qquad \forall j \in V, t \in \{1,T\}$$

$$\sum_{j\in V} f_{kj}^t = 1 \qquad \forall k \in F, t \in \{1,T\}$$

$$S_k^t = \sum_{j,l\in V} d(j,l)S_{kjl}^t \qquad \forall k \in F, t \in \{1,T\}$$

$$\sum_{j\in V} S_{kjl}^t = f_{kl}^t \qquad \forall k \in F, l \in V, t \in \{1,T\}$$

$$\sum_{l\in V} S_{kjl}^t = f_{kj}^{t-1} \qquad \forall k \in F, j \in V, t \in \{1,T\}$$

$$x_{ij}^t, f_{kj}^t, S_{klj}^t \in \{0,1\} \qquad \forall k \in F, j \in V, t \in \{1,T\}$$

Figure 6.2: Formulation of *K-facility reallocation*

leftmost position of Pos and the distance between two consecutive nodes on the path equals the distance of the respective positions on the real line. Now, the *facility reallocation problem* takes the following *discretized form*: We have a path $P = (V,E)$ that is constructed by the specific instance (x^0, C). Each facility k is initially located at a node $j \in V$ and at each stage t, each agent i is also located at a node of P. The goal is to move the facilities from node to node such that the connection cost of the agents plus the moving cost of the facilities is minimized.

To formulate this discretized version as an Integer Linear Program, we introduce some additional notation. Let $d(j,l)$ be the distance of the nodes $j,l \in V$ in P, F be the set of facilities and C be the set of agents. For each $i \in C$, $\mathrm{Loc}(i,t)$ is the node where agent i is located at stage t. We also define the following $\{0,1\}$-indicator variables for all $t \in \{1,T\}$: $x_{ij}^t = 1$ if at stage t agent i connects to a facility located at node j, $f_{kj}^t = 1$ if at stage t facility k is located at node j, $S_{kjl}^t = 1$ if facility k was at node j at stage $t-1$

and moved to node l at stage t. Now, the problem can be formulated as the Integer Linear Program depicted in Figure 6.2.

The first three constraints correspond to the fact that at every stage t, each agent i must be connected to a node j where at least one facility k is located. The constraint $\sum_{j \in V} f_{kj}^t = 1$ enforces each facility k to be located at exactly one node j. The constraint $S_k^t = \sum_{j,l \in V} d(j,l) S_{kjl}^t$ describes the cost for moving facility k from node j to node l. The final two constraints ensure that facility k moved from node j to node l at stage t if and only if facility k was at node j at stage $t-1$ and was at node l at stage t ($S_{kjl}^t = 1$ iff $f_{kl}^t = 1$ and $f_{kj}^{t-1} = 1$).

We remark that the values of f_{kj}^0 are determined by the initial positions of the facilities, which are given by the instance of the problem. The notation x_{ij}^t should not be confused with x_k^t, which is the position of facility k at stage t on the real line .

6.2.2 Rounding the Fractional Solution

Our algorithm is a simple rounding scheme of the *optimal fractional solution* of the Integer Linear Program of Figure 6.2. This simple scheme produces an integral solution that has the exact same cost with an optimal fractional solution.

Theorem 6.1. *Let x denote the solution produced by Algorithm 6.1. Then*

$$Cost(x) = \sum_{t=1}^{T} \left[\sum_{i \in C} \sum_{j \in V} d(Loc(i,t), j) x_{ij}^t + \sum_{k \in F} S_k^t \right]$$

where x_{ij}^t, S_k^t denote the values of these variables in the optimal fractional solution of the Integer Linear Program (6.2).

Theorem 6.1 is the main result of this section and it implies the optimality of our algorithm. We remind that by Lemma 6.1, there is an optimal solution that locates facilities only in positions $C_1 \cup \ldots \cup C_T \cup x^0$. This solution corresponds to an integral solution of our Integer Linear Program, meaning that $Cost(x^*)$ is greater than or equal to the cost of the *optimal fractional solution*, which by Lemma 6.1 equals $Cost(x)$. We dedicate the rest of the section to prove Theorem 6.1. The proof is conducted in two steps and each step is exhibited in Sections 6.2.3 and 6.2.4 respectively.

In section 6.2.3, we present a very simple rounding scheme in the case, where the values of the variables of the optimal fractional solution satisfy the following assumption.

Algorithm 5.1: An Optimal Algorithm for the K-Facility Reallocation

Given the initial positions $x^0 = \{x_1^0, \dots, x_K^0\}$ of the facilities and the positions of the agents $C = \{C_1, \dots, C_T\}$.

- Construct the path P and the Integer Linear Program (6.2).
- Solve the relaxation of the Integer Linear Program (6.2).
- *Rounding:* For each stage $t \geq 1$:
  - For $m = 1, \dots, K$, find the node j_m^t such that

$$\sum_{\ell=1}^{j_m^t - 1} c_\ell^t \leq m - 1 \leq \sum_{\ell=1}^{j_m^t} c_\ell^t$$

  - Locate facility m at the respective position of node j_m^t on the line

$$x_m^t \leftarrow d(j, 1) + \min_{p \in C_1 \cup \ldots \cup C_T \cup x^0} p$$

Assumption 2. *Let f_{jk}^t and c_j^t be either $1/N$ or 0, for some positive integer N.*

Although Assumption 1 is very restrictive and its not generally satisfied, it is the key step for proving the optimality guarantee of the rounding scheme presented in Algorithm 6.1. Then, in section 6.2.4 we use the rounding scheme of section 6.2.3 to prove Theorem 6.1. In the upcoming sections, $c_j^t, x_{ij}^t, f_{kj}^t, S_{kjl}^t, S_k^t$ will denote the values of these variables in the *optimal fractional* solution of the ILP (6.2).

6.2.3 Rounding Semi-Integral Solutions

Throughout this section, we suppose that Assumption 1 is satisfied; f_{kj}^t and c_j^t are either $1/N$ or 0 for some positive integer N. If the optimal fractional solution meets these requirements, then the integral solution presented in Lemma 6.2 has the same overall cost. The goal of the section is to prove Lemma 6.2.

Definition 6.3. *V_t^+ denotes the set of nodes of P with a positive amount of facility (c_j^t) at stage t,*

$$j \in V_t^+ \text{ if and only if } c_j^t > 0$$

We remind that since $c_j^t = 1/N$ or 0, $|V_t^+| = K \cdot N$. We also consider the nodes in $V_t^+ = \{Y_1^t, \dots, Y_{K\cdot N}\}$ to be ordered from left to right.

Lemma 6.2. *Let Sol be the integral solution that at each stage t places the m-th facility at the $(m-1)N+1$ node of V_t^+ i.e. $Y_{(m-1)N+1}^t$. Then, Sol has the same cost as the optimal fractional solution.*

The term **m-th facility** refers to the ordering of the facilities on the real line according to their initial positions $\{x_1^0, \dots, x_K^0\}$. The proof of Lemma 6.2 is quite technically complicated, however it is based on two intuitive observations about the optimal fractional solution.

Observation 6.1. *The set of nodes at which **agent** i **connects** at stage t are **consecutive** nodes of V_t^+. More precisely, there exists a set $\{Y_\ell^t, \dots, Y_{\ell+N-1}^t\} \subseteq V_t^+$ such that*

$$\sum_{j \in V} d(Loc(i,t), j) x_{ij}^t = \frac{1}{N} \sum_{h=\ell}^{\ell+N-1} d(Loc(i,t), Y_h^t)$$

Proof. Let an agent i that at some stage t has $x_{iY_j^t}^t > 0, x_{iY_\ell^t}^t < 1/N$ and $x_{iY_h^t}^t > 0$ for some $j < \ell < h$. Assume that $\text{Loc}(i,t) \le Y_\ell^t$ and to simplify notation consider $x_\ell = x_{iY_\ell^t}^t, x_h = x_{iY_h^t}^t$. Now, increase x_ℓ by ϵ and decrease x_h by ϵ, where $\epsilon = \min(1/N - x_\ell, x_h)$. Then, the cost of the solution is decreased by $(d(\text{Loc}(i,t), h) - d(\text{Loc}(i,t), \ell))\epsilon > 0$, thus contradicting the optimality of the solution. The same argument holds if $\text{Loc}(i,t) \ge Y_\ell^t$. The proof follows since $\sum_{j \in V} x_{ij}^t = 1$. □

Observation 6.2. *Under Assumption 1, the m-th facility places amount of facility $f_{mj}^t = 1/N$ from the $(m-1)N+1$ to the mN node of V_t^+ i.e. to nodes $\{Y_{(m-1)N+1}^t, \dots, Y_{mN}^t\}$.*

Observation 6.2 serves in understanding the structure of the optimal fractional solution under Assumption 1. However, it will be not used in this form in the rest of the section. We use Lemma 6.3 instead, which is roughly a different wording of Observation 6.2 and its proof can be found in subsection 6.3 at the end of the section.

Lemma 6.3. *Let S_k^t the fractional moving cost of facility k at stage t. Then*

$$\sum_{t=1}^{T} \sum_{k \in F} S_k^t = \frac{1}{N} \sum_{t=1}^{T} \sum_{j=1}^{K\cdot N} d(Y_j^{t-1}, Y_j^t)$$

Observations 6.1, and Lemma 6.3 (Observation 6.2) are the key points in proving Lemma 6.2.

Definition 6.4. *Let Sol_p be the integral solution that places at stage t the m-th facility at the $(m-1)N+p$ node of V_t^+ i.e. $Y^t_{(m-1)N+p}$.*

Notice that the integral solution Sol referred in Lemma 6.2 corresponds to Sol_1. The proof of Lemma 6.2 follows directly by Lemma 6.4 and Lemma 6.5 that conclude this section.

Lemma 6.4. *Let S_k^t be the moving cost of facility k at stage t in the optimal fractional solution and let $MovingCost(Sol_p)$ be the total moving cost of the facilities in the integral solution Sol_p. Then,*

$$\frac{1}{N}\sum_{p=1}^{N} MovingCost(Sol_p) = \sum_{t=1}^{T}\sum_{k\in F} S_k^t$$

Proof. By the definition of the solutions Sol_p we have that:

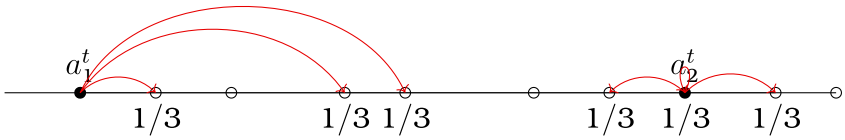
$$\begin{aligned}
\frac{1}{N}\sum_{p=1}^{N} MovingCost(Sol_p) &= \frac{1}{N}\sum_{p=1}^{N}\sum_{t=1}^{T}\sum_{m=1}^{K} d(Y^{t-1}_{(m-1)N+p}, Y^t_{(m-1)N+p}) \\
&= \frac{1}{N}\sum_{t=1}^{T}\sum_{m=1}^{K}\sum_{p=1}^{N} d(Y^{t-1}_{(m-1)N+p}, Y^t_{(m-1)N+p}) \\
&= \frac{1}{N}\sum_{t=1}^{T}\sum_{j=1}^{K\cdot N} d(Y^{t-1}_j, Y^t_j) \\
&= \sum_{t=1}^{T}\sum_{k\in F} S_k^t
\end{aligned}$$

The last equality comes from Lemma 6.3. □

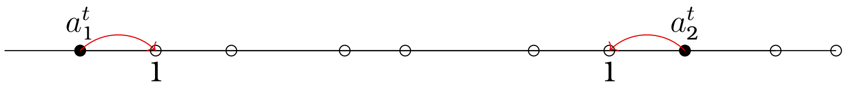
Lemma 6.4 states that if we pick uniformly at random one of the N integral solutions $\{Sol_p\}_{p=1}^N$, then the expected moving cost that we will pay is equal to the moving cost paid by the optimal fractional solution. Interestingly, the same holds for the expected connection cost. This is formally stated in Lemma 6.5 and it is where Observation 6.1 comes into play.

Lemma 6.5. *Let $ConCost_i^t(Sol_p)$ denote the connection cost of agent i at stage t in Sol_p. Then,*

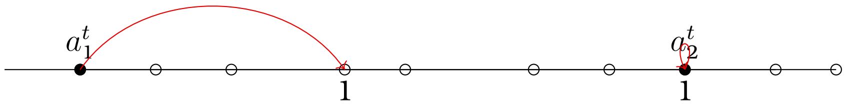
$$\frac{1}{N}\sum_{p=1}^{N} ConCost_i^t(Sol_p) = \sum_{j\in V} d(Loc(i,t), j)x_{ij}^t$$



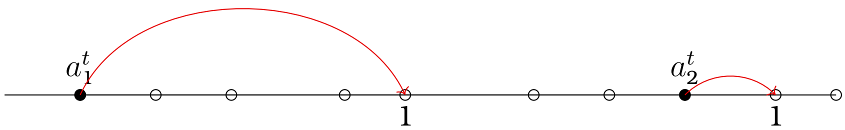
Z^*_{LP} opens at each node amount of facility $c_j^t = 1/3$ or 0 ($N = 3$ and $K = 2$). The set Y^t is the set of nodes with $c_j^t = 1/3$. The red acres show the connection cost that agents 1 and 2 suffer respectively.



Since $N = 3$ and $|Y^t| = 6$. Sol_1 opens facilites in Y_1^t and Y_4^t. The red acres show the connection cost that agents 1 and 2 suffer respectively in Sol_1.



Since $N = 3$ and $|Y^t| = 6$. Sol_2 opens facilites in Y_2^t and Y_5^t. The red acres show the connection cost that agents 1 and 2 suffer respectively in Sol_2.



Since $N = 3$ and $|Y^t| = 6$. Sol_3 opens facilites in Y_3^t and Y_6^t. The red acres show the connection cost that agents 1 and 2 suffer respectively in Sol_3.

Figure 6.3: In the depicted instance $N = 3$ and $K = 2$. The figure illustrates the positions in which Sol_1, Sol_2 and Sol_3 of Definition 6.4 open facilities. One can also easily verify Lemma 6.5.

As already mentioned, the proof of Lemma 6.5 crucially makes use of Observation 6.1 and is presented in the subsection 6.3 at the end of the section. Combining Lemma 6.4 and 6.5 we get that if we pick an integral solution Sol_p uniformly at random, the average total cost that we pay is Z^*_{LP}, where Z^*_{LP} is the optimal fractional cost. More precisely,

$$\begin{aligned}
\frac{1}{N}\sum_{p=1}^{N} Cost(Sol_p) &= \frac{1}{N}\sum_{p=1}^{N}[MovingCost(Sol_p) + \sum_{t=1}^{T}\sum_{i\in C} ConCost_i^t(Solp)] \\
&= \sum_{t=1}^{T}[\sum_{k=1}^{K} S_k^t + \sum_{i\in C}\sum_{j\in V} d(\text{Loc}(i,t),j)x_{ij}^t] \\
&= Z^*_{LP}
\end{aligned}$$

Since $Sol_p \geq Z^*_{LP}$, we have that $Sol_1 = \cdots = Sol_N = Z^*_{LP}$ and this proves Lemma 6.2.

6.2.4 Rounding the General Case

In this section we use Lemma 6.2 to prove Theorem 6.1. As already discussed, Assumption 1 is not satisfied in general by the fractional solution of the linear program (6.2). Each $S^t_{kj\ell}$ will be either 0 or $A^t_{kj\ell}/N^t_{kj\ell}$ for positive some integers $A^t_{kj\ell}, N^t_{kj\ell}$. Moreover each positive f^t_{kj} will have the form B^t_{kj}/N, where $N = \Pi_{S^t_{kj\ell}>0} N^t_{kj\ell}$. and this is due to the constraint $f^t_{kj} = \sum_{j\in V} S^t_{kj\ell}$.

Consider the path $P' = (V', E')$ constructed from path $P = (V, E)$ as follows: Each node $j \in V$ is splitted into $K \cdot N$ *copies* $\{j_1, \dots, j_{KN}\}$ with zero distance between them. Consider also the LP (6.2), when the underlying path is $P' = (V', E')$ and at each stage t, each agent i is located to a node of V' that is a *copy* of i's original location, $Loc'(i,t) = \ell \in V'$ where $\ell \in \text{Copies}(\text{Loc}(i,t))$. Although these are two different LP's, they are closely related since a solution for the one can be converted to a solution for the other with the exact same cost. This is due to the fact that for all $j, h \in V$, $d(j,h) = d(j', h')$ for $j' \in \text{Copies}(j)$ and $h' \in \text{Copies}(h)$.

The reason that we defined P' and the second LP is the following: Given an optimal fractional solution of the LP defined for P, we will construct a fractional solution for the LP defined for P' with the exact same cost, which additionally satisfies Assumption 1. Then, using Lemma 6.2 we can obtain an integral solution for P' with the same cost. This integral solution for P' can be easily converted to an integral solution for P. We finally show that these steps are done *all at once* by the rounding scheme of Algorithm 6.1 and this concludes the proof of Theorem 6.1.

Given the fractional positions $\{f^t_{kj}\}_{t\geq 1}$ of the optimal solution of the LP formulated for $P = (V, E)$, we construct the fractional positions of the facilities in $P' = (V', E')$ as follows: If $f^t_{kj} = B^t_{kj}/N$, then facility k puts a $1/N$ amount of facility in B^t_{kj} nodes of the set $\text{Copies}(j) = \{j_1, \dots, j_{KN}\}$ that have a 0 amount of facility. The latter is possible since there are exactly $K \cdot N$ *copies* of each $j \in V$ and $c^t_j \leq K$ (that is the reason we required $K \cdot N$ copies of each node). The values of the rest of the variables are defined in the proof of Lemma 6.7 that is presented in the end of the section. The key point is that the produced solution $\{f'^t_{k\ell}, c'^t_j, S'^t_{kj\ell}, x'^t_{ij}, S'^t_k\}$ will satisfy the following properties (see Lemma 6.7):

- its cost equals Z^*_{LP}
- $f'^t_{k\ell} = 1/N$ or 0, for each $\ell \in V'$

- $c'^t_\ell = 1/N$ or 0, for each $\ell \in V'$
- $c^t_j = \sum_{\ell \in \text{Copies}(j)} c'^t_\ell$, for each $j \in V$

Clearly, this solution satisfies Assumption 1 and thus Lemma 6.2 can be applied. This implies that the integral solution for P' that places the m-th facility to the $(m-1)N+1$ node of V'^+_t ($Y'^t_{(m-1)N+1} \in V'$) has cost Z^*_{LP}. So the integral solution for P that places the m-th facility to the node $j^t_m \in V$, such that $Y'^t_{(m-1)N+1} \in \text{Copies}(j^t_m)$, has again cost Z^*_{LP}.

A naive way to determine the nodes j^t_m is to calculate N, construct P' and its fractional solution, find the nodes $Y'^t_{(m-1)N+1}$ and determine the nodes j^t_m of P. Obviously, this rounding scheme requires exponential time. Fortunately, Lemma 6.6 provides a linear time rounding scheme to determine the node j^t_m given the optimal fractional solution of $P = (V, E)$. This concludes the proof of Theorem 6.1.

Lemma 6.6. *The* $(m-1)N+1$ *node of* V'^+_t *is a* copy *of the node* $j^t_m \in V$ *if and only if*

$$\sum_{\ell=1}^{j^t_m - 1} c^t_\ell \leq m-1 < \sum_{\ell=1}^{j^t_m} c^t_\ell$$

.

Proof. Let $(m-1)N+1$ node of V'^+_t be a *copy* of the node $j^t_m \in V^+_t$. Then

$$\sum_{\ell=1}^{j^t_m - 1} c^t_\ell = \sum_{\ell=1}^{j^t_m - 1} \sum_{\ell' \in \text{Copies}(\ell)} c'^t_{\ell'} \leq (m-1)N\frac{1}{N} = m-1$$

$$\sum_{\ell=1}^{j^t_m} c^t_\ell = \sum_{\ell=1}^{j^t_m} \sum_{\ell' \in \text{Copies}(\ell)} c'^t_{\ell'} = ((m-1)N+1)\frac{1}{N} > m-1$$

The above equations hold because of the property $c^t_\ell = \sum_{\ell' \in \text{Copies}(\ell)} c'^t_{\ell'}$ and that $c'^t_{\ell'}$ is either 0 or $1/N$.
Now, let $\sum_{\ell=1}^{j^t_m - 1} c^t_\ell \leq m-1 < \sum_{\ell=1}^{j^t_m} c^t_\ell$ and assume that $(m-1)N+1$-th node of V^+_t is a copy of $j \in V$. If $j < j^t_m$, then $\sum_{\ell=1}^{j} c^t_\ell > m-1$ and if $j > j^t_m$, then $\sum_{\ell=1}^{j^t_m} < m-1$. As a result, $j = j^t_m$. □

Lemma 6.7. *Let* $\{f^t_{kj}, c^t_j, S^t_{kjl}, x^t_{ij}\}_{t \geq 1}$ *the optimal fractional solution for the LP 6.2 with underlying path* P. *Then, there exists a solution* $\{f'^t_{kj}, c'^t_j, S'^t_{kjl}, x'^t_{ij}, S'^t_k\}_{t \geq 1}$ *of the LP 6.2 with underlying path* P' *such that*

1. *Its cost is* Z^*_{LP}.

2. $f'^t_{k\ell} = 1/N$ *or* 0*, for each* $\ell \in V'$

3. $c'^t_{\ell} = 1/N$ *or* 0*, for each* $\ell \in V'$

4. $c^t_j = \sum_{\ell \in Copies(j)} c'^t_{\ell}$*, for each* $j \in V$

Proof. First, we set values to the variables f'^t_{kj}. Initially, all $f'^t_{kj} = 0$. We know that if $f^t_{kj} > 0$, then it equals B^t_{kj}/N, for some positive integer B^t_{kj}. For each such f^t_{kj}, we find $u_1, \ldots, u_{B^t_{kj}} \in \text{Copies}(j)$ with $f'^t_{ku_h} = 0$. Then, we set $f'^t_{ku_h} = 1/N$ for $h = \{1, B^t_{kj}\}$. Since there are KN copies of each node $j \in V$ and $\sum_{j \in V} f^t_{kj} \leq K$, we can always find sufficient copies of j with $f'^t_{ku} = 0$. When this step is terminated, we are sure that conditions $2, 3, 4$ are satisfied.

We continue with the variables $S'^t_{kj\ell}$. Initially, all $S'^t_{kj\ell} = 0$. Then, each positive $S^t_{kj\ell}$ has the form $B^t_{kj\ell}/N$. Let $B = B^t_{kj\ell}$ to simplify notation. We now find B copies of $u_1, \ldots, u_B$ of j and $v_1, \ldots, v_B$ of ℓ so that

- $f'^t_{ku_1} = \cdots = f'^t_{ku_B} = f'^t_{kv_1} = \cdots = f'^t_{kv_B} = 1/N$
- $S'^t_{ku_1h} = \cdots = S'^t_{ku_Bh} = S'^t_{khv_1} = \cdots = S'^t_{khv_B} = 0$ for all $h \in V'$

We then set $S'^t_{ku_1v_1} = \cdots = S'^t_{ku_Bv_B} = 1/N$. Again, since $\sum_{\ell \in V} S^t_{kj\ell} = f^t_{kj}$ and $\sum_{j \in V} S^t_{kj\ell} = f^t_{k\ell}$ we can always find $B^t_{kj\ell}$ pairs of copies of j and ℓ that satisfy the above requirements. We can now prove that the movement cost of each facility k is the same in both solutions.

$$\begin{aligned}
\sum_{j \in V} \sum_{\ell \in V} d(j,\ell) S^t_{kj\ell} &= \sum_{j \in V} \sum_{\ell \in V} d(j,\ell) B^t_{kj\ell}/N \\
&= \sum_{j \in V} \sum_{\ell \in V} \sum_{h \in \text{Copies}(j)} \sum_{h' \in \text{Copies}(\ell)} S'^t_{khh'} d(h,h') \\
&= \sum_{j' \in V'} \sum_{\ell' \in V'} S'^t_{kj'\ell'} d(j',\ell')
\end{aligned}$$

The second equality follows from the fact that h, h' are copies of j, ℓ respectively and thus $d(h,h') = d(j,\ell)$.

Finally, set values to the variables x'^t_{ij} for each $j \in V'$. Again, each positive x^t_{ij} equals B^t_{ij}/N, for some positive integer. We take B^t_{ij} copies of j, $u_1, \ldots, u_{B^t_{ij}}$ and set $x'^t_{iu_1} = \cdots = x'^t_{iu_{B^t_{ij}}} = 1/N$. The connection cost of each agent i remains the same since

$$\begin{aligned}\sum_{j\in V} d(\mathrm{Loc}(i,t),j)x_{ij}^t &= \sum_{j\in V} d(\mathrm{Loc}(i,t),j)B_{ij}^t/N \\ &= \sum_{j\in V} d(\mathrm{Loc}(i,t),j)\sum_{j'\in \mathrm{Copies}(j)} x'^t_{ij'} \\ &= \sum_{j\in V}\sum_{j'\in \mathrm{Copies}(j)} d(\mathrm{Loc}'(i,t),j')x'^t_{ij'} \\ &= \sum_{h\in V'} d(\mathrm{Loc}'(i,t),h)x'^t_{ih}\end{aligned}$$

The third equality holds since $\mathrm{Loc}'(i,t) \in \mathrm{Copies}(\mathrm{Loc}(i,t))$. □

6.3 Omitted proofs

Lemma 6.3 *Let S_k^t the fractional switching cost of facility k at stage t. Then,*

$$\sum_{t=1}^{T}\sum_{k\in F} S_k^t = \frac{1}{N}\sum_{t=1}^{T}\sum_{j=1}^{K\cdot N} d(Y_j^{t-1},Y_j^t)$$

Proof. By Assumption 1, $c_j^t = 1/N$ if $j \in V_t^+ = \{Y_1^t,\ldots,Y_{KN}^t\}$ and 0 otherwise. Notice that the connection cost of the optimal fractional solution only depends on the variables c_j^t. As a result, $f_{kj}^t, S_k^t, S_{kjl}^t$ must be the optimal solution of the following linear program.

$$\begin{aligned}
&\text{minimize} && \sum_{t=1}^{T}\sum_{k=1}^{K} S_k^t \\
&\text{s.t.} && \sum_{k\in F} f_{kj}^t = \tfrac{1}{N} && \forall j \in V_t^+, t\in\{1,T\}\\
& && \sum_{j\in V_t^+} f_{kj}^t = 1 && \forall k\in F, t\in\{1,T\}\\
& && S_k^t = \sum_{j,l\in V} d(j,l)S_{kjl}^t && \forall k\in F, t\in\{1,T\}\\
& && \sum_{j\in V_{t-1}^+} S_{kjl}^t = f_{kl}^t && \forall k\in F, l\in V_t^+, t\in\{1,T\}\\
& && \sum_{l\in V_t^+} S_{kjl}^t = f_{kj}^{t-1} && \forall k\in F, j\in V_{t-1}^+, t\in\{1,T\}
\end{aligned}$$

Instead of proving that the minimum cost of the above linear program is $\frac{1}{N}\sum_{t=1}^{T}\sum_{j=1}^{K\cdot N} d(Y_j^{t-1},Y_j^t)$, we prove this for the following more convenient relaxation of the above LP.

$$\begin{aligned}
&\text{minimize} && \sum_{t=1}^{T} \sum_{j\in V_{t-1}^{+}, l\in V_t^{+}} d(j,l)F_{jl}^{t} && \\
&\text{s.t.} && \sum_{l\in V_t^{+}} F_{jl}^{t} = \tfrac{1}{N} && \forall j \in V_{t-1}^{+}, t\in\{1,T\} \\
& && \sum_{j\in V_{t-1}^{+}} F_{jl}^{t} = \tfrac{1}{N} && \forall l \in V_{t}^{+}, t\in\{1,T\}
\end{aligned} \tag{6.1}$$

It is easy to prove that the LP (6.1) is a relaxation of the first by setting $F_{jl}^t = \sum_{k\in F} S_{kjl}^t$. Moreover, the above LP describes a flow problem between the nodes V_t^+, where F_{jl}^t is the amount of flow going from node $j \in V_{t-1}^+$ to node $l \in V_t^+$ (see Figure 6.4).

We are ready for the final step of our proof. First, observe that $F^t_{Y_j^{t-1}Y_j^t}$ is feasible solution for the above LP since $|V_{t-1}^+| = |V_t^+| = K \cdot N$. If we prove that this assignment minimizes the objective, then we are done. Assume that in the optimal solution, $F^t_{Y_1^{t-1}Y_1^t} < 1/N$. Since $\sum_{l\in V_t^+} F^t_{Y_1^{t-1}l} = \frac{1}{N}$, there exists Y_j^t such that $F^t_{Y_1^{t-1}Y_j^t} > 0$. Similarly, by using the second constraint we obtain that $F^t_{Y_{j'}^{t-1}Y_1^t} > 0$. Let $\epsilon = \min(F^t_{Y_1^{t-1}Y_j^t}, F^t_{Y_{j'}^{t-1}Y_1^t})$. Observe that if we increase $F^t_{Y_1^{t-1}Y_1^t}$, $F^t_{Y_{j'}^{t-1}Y_j^t}$ by ϵ and decrease $F^t_{Y_1^{t-1}Y_j^t}$, $F^t_{Y_{j'}^{t-1}Y_1^t}$ by ϵ, we obtain another feasible solution. The cost difference of the two solutions is $D = \epsilon(d(Y_1^{t-1}, Y_j^t) + d(Y_{j'}^{t-1}, Y_1^t) - d(Y_1^{t-1}, Y_1^t) - d(Y_{j'}^{t-1}, Y_j^t))$. If we prove that D is no negative, we are done. We show the latter using the fact that $Y_1^{t-1} \leq Y_{j'}^{t-1}$ and $Y_1^t \leq Y_j^t$. More precisely,

- If $Y_1^{t-1} \leq Y_1^t$ then $D \geq 0$ since $Y_1^t \leq Y_j^t$.
- If $Y_1^{t-1} \geq Y_1^t$ then $D \geq 0$ since $Y_1^{t-1} \leq Y_{j'}^{t-1}$.

Until now, we have shown that in the optimal solution, the node Y_1^{t-1} sends all of her flow to the node Y_1^t. Meaning that Y_1^t does not receive flow by any other node apart from Y_1^{t-1}. By repeating the same argument, it follows that in the optimal solution each node Y_j^{t-1} sends all of her flow to Y_j^t. □

Lemma 6.5 *Let $ConCost_i^t(Sol_p)$ denote the connection cost of agent i at stage t in Sol_p of Definition 6.4. Then*

$$\frac{1}{N}\sum_{p=1}^{N} ConCost_i^t(Sol_p) = \sum_{i\in C}\sum_{j\in V} d(\mathrm{Loc}(i,t),j)x_{ij}^t$$

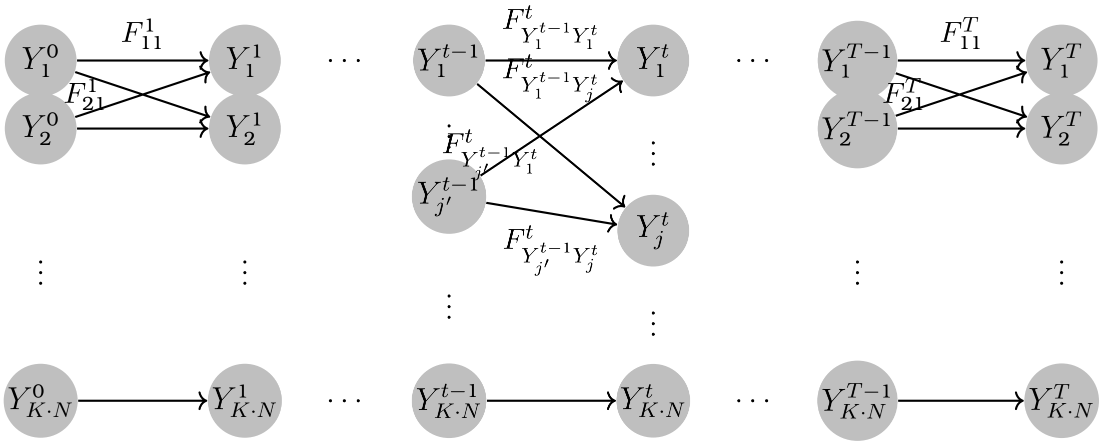


Figure 6.4: The flow described by LP (6.1).

Proof. We will prove that $\frac{1}{N}\sum_{p=1}^{N} \text{ConCost}_i^t(Sol_p)$ equals $\sum_{j\in V} d(\text{Loc}(i,t),j)x_{ij}^t$. We remind that by Assumption 1, c_j is $1/N$ if $j \in V_t^+$ and 0 otherwise. As a result, in the optimal fractional solution, each agent i finds the N closest to $\text{Loc}(i,t)$ nodes of V_t^+ and receives a $1/N$ amount of service from each one of them. Let us call this set N_i^t. By Observation 6.1, the nodes in N_i^t must be consecutive nodes of V_t^+ i.e. $N_i^t = \{Y_l^t, \ldots, Y_{l+N-1}^t\}$ and

$$\sum_{j\in V} d(\text{Loc}(i,t),j)x_{ij}^t = \sum_{j=l}^{l+N-1} d(\text{Loc}(i,t),Y_j^t)/N$$

Since Sol_p puts facilities in the positions $\{Y_{(m-1)\cdot N+p}^t\}_{m=1}^K$, there exists a unique node $Y_{l(p)}^t \in N_i^t$ in which Sol_p puts a facility. $Y_{l(p)}^t$ is the closest node to $\text{Loc}(i,t)$ from all the nodes in which Sol_p puts a facility. As a result, $ConCost_i^t(Sol_p) = d(\text{Loc}(i), Y_{l(p)}^t)$. Now, summing over p we get,

$$\begin{aligned}\frac{1}{N}\sum_{p=1}^{N} ConCost_i^t(Sol_p) &= \frac{1}{N}\sum_{p=1}^{N} d(\text{Loc}(i),Y_{l(p)}^t)\\ &= \sum_{j=l}^{l+N-1} d(\text{Loc}(i),Y_j^t)/N\\ &= \sum_{j\in V} d(\text{Loc}(i,t),j)x_{ij}^t\end{aligned}$$

□

Chapter 7

Open Problems

In this chapter we list several interesting open problems that came out during this thesis and we did not manage to solve yet. I hope that all of these question will meet their answers in the near future.

A first interesting question left open by this thesis concerns the computational complexity of finding Nash Equilibrium in *coevolutionary opinion formation games* introduced in [15]. In this kind of opinion formation games the weights measuring the influence among the agents are not static, but depend on the agents' expressed opinions. The existence of Nash Equilibrium in coevolutionary opinion formation games is guaranteed by the *Kakutani Fixed Point Theorem* [15, 126] and thus finding one belongs in the *Polynomial Parity Arguments on Directed graphs* (PPAD) complexity class [120]. Determining the computational complexity class for which this problem is complete has not yet received an answer, while my conjecture is that this problem is not PPAD-complete.

A similar open question concerns the computational complexity of finding equilibria in the FJ model with negative weights among the agents. Although our results presented in Chapter 4 provide an illustrative picture on the cases in which simultaneous best response dynamics converges, they do not have much to say about the cases in which finding an equilibrium is computationally easy. In its general form this problem is hard (and this indicates an additional reason for the assumption introduced in Chapter 4) since the PLS-complete problem Local-MaxCut [130] can be very easily reduced to computing Nash Equilibrium in instances where *all* the weights among the agents are negative. However the mild assumption that the sum of the weights of each agent is positive breaks down this computational hardness. The reason is that the latter assumption makes the agents' disagreement cost functions convex and thus the existence of equilibrium is implied by the *Kakutani Fixed Point Theorem* [126]. As a result, computing Nash Equilibrium in opinion formation games in which the above assumption holds, renders the problem at the interesection of PPAD and PLS complexity classes [57]. I am very intrigued towards understanding whether this problem can be solved in polynomial time or it is complete in a complexity class, such as the *Continous Local Search class* (CLS) [57], contained inside the intersection of PPAD and PLS.

Another question that I have tried to answer concerns the convergence rate of Network HK model. Our convergence results presented in Chapter 5 are asymptotic in the sense that guarantee that at some point in time the overall system freezes, but the do not provide any kind of guarantees on the number of steps needed for this to happen. I conjecture that the convergence time of Network HK is polynomially bounded by the number of agents, something that is also indicated by our experimental evaluations.

A final problem that is left open by this thesis, concerns the online version of the k-facility reallocation problem examined in Chapter 6. In the online version of the problem the requests of the clients at each round are revealed only after the determinations of the positions of the facilities at the previous round. With a quite easy argument, one can prove that there is no deterministic algorithm that can approximate the optimal solution with a factor smaller than $\Theta(k)$. We strongly believe that this bound is tight, but we have not yet managed to find a k-competitive algorithm for this problem.

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