National Technical University of Athens
School of Electrical and Computer Engineering
Division: **Information Transmission Systems and Material Technology**

Black Holes in the GW Detection Era and the Role of Symmetries

Diploma Thesis

Gounis Leonidas–Rafail

Supervisor: George Fikioris
Professor, National Technical University of Athens

Athens, November 2025

National Technical University of Athens

School of Electrical and Computer Engineering

Division: **Information Transmission Systems and Material Technology**

Black Holes in the GW Detection Era and the Role of Symmetries

Diploma Thesis

Gounis Leonidas–Rafail

Supervisor: George Fikioris
Professor, National Technical University of Athens

Approved by the three-member scientific committee on the ***11 November 2025***.

....	....	....
George Fikioris Professor, N.T.U.A.	Alexandros Kehagias Professor, N.T.U.A.	Christos Tsironis Professor, N.T.U.A.

Athens, November 2025

Gounis Leonidas–Rafail

Graduate of the School of Electrical and Computer Engineering,
National Technical University of Athens

Περίληψη

Τα βαρυτικά κύματα και οι μαύρες τρύπες αποτελούν θεμελιώδεις προβλέψεις της Γενικής Θεωρίας της Σχετικότητας, οι οποίες έχουν επιβεβαιωθεί από πρωτοποριακές παρατηρήσεις, όπως η ανίχνευση βαρυτικών κυμάτων από συγχωνεύσεις μαύρων τρυπών από τις συνεργασίες *LIGO* και *Virgo*. Κατά τη φάση της σπειροειδούς σύγκλισης ενός συμπαγούς διπλού συστήματος, όπως αυτά που περιλαμβάνουν αστέρες νετρονίων ή μαύρες τρύπες, οι παλιρροϊκές αλληλεπιδράσεις γίνονται σημαντικές όταν η τροχιακή απόσταση γίνει επαρκώς μικρή. Αυτά τα παλιρροϊκά φαινόμενα χαρακτηρίζονται από παραμέτρους γνωστές ως αριθμοί *Love*, οι οποίοι ποσοτικοποιούν την παραμόρφωση ενός αντικειμένου ως απόκριση στο βαρυτικό πεδίο του συνοδού του. Ειδικότερα, οι στατικοί παλιρροϊκοί αριθμοί *Love* εξαρτώνται από την εσωτερική δομή και τη σύσταση των συμπαγών αντικειμένων που υφίστανται παλιρροϊκή παραμόρφωση. Αντίθετα, για τις μαύρες τρύπες αναμένεται ότι οι *TLNs* είναι μηδενικοί, λόγω της απουσίας άκαμπτης υλικής δομής. Στην παρούσα διπλωματική εργασία, εξετάζουμε το ζήτημα της μηδενικότητας των στατικών *TLNs* των μαύρων τρυπών *Kerr*, χρησιμοποιώντας τον φορμαλισμό *Ernst* και συντεταγμένες *Weyl* για να αναλύσουμε την παλιρροϊκή απόκρισή τους. Το αποτέλεσμα αυτό αναδεικνύει τη στιβαρότητα των επιχειρημάτων που βασίζονται σε συμμετρίες και τα οποία διέπουν την απόκριση των μαύρων τρυπών, και υπογραμμίζει τον ιδιαίτερο χαρακτήρα τους ως λύσεων της Γενικής Σχετικότητας. Οι μηδενικοί *TLNs* επιβεβαιώνουν την αρχή ότι οι μαύρες τρύπες, σε αντίθεση με άλλα συμπαγή αντικείμενα, δεν διατηρούν καμία μόνιμη παραμόρφωση υπό την επίδραση στατικών παλιρροϊκών δυνάμεων. Η μελέτη αυτή συμβάλλει στη βαθύτερη κατανόηση της φυσικής των μαύρων τρυπών, προσφέροντας νέες οπτικές για την αλληλεπίδρασή τους με εξωτερικά πεδία και για τις συνέπειες στην αστρονομία βαρυτικών κυμάτων.

Λέξεις κλειδία Μαύρες Τρύπες, Αριθμοί Λοβ, Γενική Σχετικότητα, Άινστάιν.

Abstract

Gravitational Waves (GWs) and Black Holes (BHs) are key predictions of General Relativity (GR), validated by groundbreaking observations such as the detection of GWs from BH mergers by the LIGO and Virgo collaborations. During the inspiral phase of a compact binary system, such as those involving neutron stars or BHs, tidal interactions become significant when the orbital separation is sufficiently small. These tidal effects are characterized by parameters known as Love Numbers, which quantify an object's deformation in response to the gravitational field of its companion. In particular the static Tidal Love Numbers (TLNs) depend on the internal structure and composition of the compact objects undergoing tidal deformation. In contrast, BHs are expected to have zero TLNs due to their lack of a rigid structure. In this thesis, we address this question of the vanishing of the static TLN of Kerr BHs by employing the Ernst formalism and Weyl coordinates to analyze the tidal response of Kerr BHs. This result highlights the robustness of the symmetry-based arguments that govern BH responses and underscores the distinctive nature of BHs as solutions to GR. The vanishing TLNs reaffirm the principle that BHs, unlike other compact objects, do not retain any permanent deformation under static tidal forces. This study contributes to the broader understanding of BH physics, offering new perspectives on their interaction with external fields and implications for gravitational wave astronomy.

Keywords: Einstein, General Realtivity, Black Holes, Love Numbers.

Ευχαριστίες

Καθώς ολοκληρώνεται η φοίτησή μου στη Σχολή Ηλεκτρολόγων Μηχανικών και Μηχανικών Υπολογιστών του Εθνικού Μετσόβιου Πολυτεχνείου, θα ήθελα αρχικά να ευχαριστήσω θερμά τον Καθηγητή κ. Αλέξανδρο Κεχαγιά για την εμπιστοσύνη που μου έδειξε, επιτρέποντάς μου να εκπονήσω την εργασία αυτή υπό την επίβλεψή του και για το ενδιαφέρον που μου καλλιέργησε για την Θεωρητική και Μαθηματική Φυσική. Θα ήθελα επίσης εκφράσω την ειλικρινή μου ευγνωμοσύνη προς τους συν-επιβλέποντες μου, τους Καθηγητές κ. Γεώργιο Φικιόρη και τον κ. Χρήστο Τσιρώνη αλλά και στους Κ. Αναγνωστόπουλο, Χ. Κούβαρη και Ν. Μαυρώματο για την ανιδιοτελή βοήθεια και καθοδήγηση που μου παρείχαν όλα αυτά τα χρόνια. Η βοήθειά τους υπήρξε πολύτιμη, τόσο κατά την εκπόνηση της παρούσας πτυχιακής εργασίας όσο και μέσα από τις συμβουλές τους για το μέλλον μου. Σημαντικότεροι συνοδοιπόροι μου σε αυτό το ακαδημαϊκό ταξίδι, όπως και σε όλα μου τα βήματα, ήταν η οικογένειά μου και γι΄ αυτό τους ευχαριστώ θερμά. Τέλος, θα ήθελα να ευχαριστήσω τον Πέτρο, τον Μάθιου, το Μιχάλη και όλους τους φίλους μου για τις όμορφες αναμνήσεις και τη συντροφιά τους.

Contents

List of Tables

List of Figures

Κεφάλαιο 1

Εκτεταμένη περίληψη στα Ελληνικά

1.1 Η ειδική θεωρία της σχετικότητας

1. Η ιδέα της σχετικότητας στον Γαλιλαίο και το κλασικό πλαίσιο

Πολύ πριν από την Ειδική και τη Γενική Σχετικότητα, ο Γαλιλαίος εντόπισε μια βαθιά αμεταβλητότητα της φύσης: αν κλειστούμε σε μια καμπίνα πλοίου που κινείται ευθύγραμμα και ομαλά, κανένα αποκλειστικά μηχανικό πείραμα εντός της καμπίνας δεν αποκαλύπτει αν το πλοίο είναι ακίνητο ή κινείται. Από αυτό το νοητικό πείραμα αποστάλαξε την *γαλιλαϊκή αρχή της σχετικότητας*: όλοι οι μηχανικοί νόμοι έχουν την ίδια μορφή σε εργαστήρια που κινούνται με σταθερή σχετική ταχύτητα.

Δεύτερος πυλώνας είναι ο *νόμος της αδράνειας*: σώμα χωρίς εξωτερικές δυνάμεις παραμένει ακίνητο ή κινείται ευθύγραμμα με σταθερή ταχύτητα. Ο Νεύτων τον υιοθέτησε ως πρώτο νόμο και συνέδεσε τη δυναμική με την έννοια του *αδρανειακού συστήματος αναφοράς*, όπου ισχύει η αδράνεια. Στο προ-σχετικιστικό πλαίσιο ο χρόνος θεωρείται απόλυτος και ο χώρος ευκλείδειος· όλοι οι αδρανειακοί παρατηρητές συμφωνούν για χρονικά διαστήματα και χωρικές αποστάσεις όταν μετρώνται στο ίδιο σύστημα.

Θεωρούμε δύο αδρανειακά συστήματα S και S' με σχετική ταχύτητα V κατά x. Στη νευτώνεια κινηματική οι συντεταγμένες συνδέονται με τους *μετασχηματισμούς Γαλιλαίου*:

$$x' = x - Vt, \qquad y' = y, \qquad z' = z, \qquad t' = t,$$

οι οποίοι διατηρούν τη μορφή των εξισώσεων του Νεύτωνα. Η ταχύτητα μετασχηματίζεται ως $\boldsymbol{v}' = \boldsymbol{v} - \boldsymbol{V}$ και η επιτάχυνση είναι αμετάβλητη ($\boldsymbol{\alpha}' = \boldsymbol{\alpha}$), εξηγώντας γιατί η δυναμική $m\boldsymbol{a} = \boldsymbol{F}$ είναι μορφοαναλλοίωτη μεταξύ αδρανειακών πλαισίων. Στην πράξη, αυστηρά αδρανειακά πλαίσια δεν υπάρχουν (π.χ. γήινο εργαστήριο), αλλά συχνά υπάρχουν πολύ καλές προσεγγίσεις (ηλιοκεντρικό πλαίσιο).

2. Η ηλεκτρομαγνητική θεωρία και η πρώτη ένδειξη κρίσης

Οι εξισώσεις του $Maxwell$ στην κενότητα

$$\nabla \cdot \boldsymbol{E} = 0, \qquad \nabla \cdot \boldsymbol{B} = 0, \qquad \nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}, \qquad \nabla \times \boldsymbol{B} = \mu_0 \varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t},$$

οδηγούν σε κυματικές εξισώσεις

$$\nabla^2 \boldsymbol{E} = \frac{1}{c^2}\frac{\partial^2 \boldsymbol{E}}{\partial t^2}, \qquad \nabla^2 \boldsymbol{B} = \frac{1}{c^2}\frac{\partial^2 \boldsymbol{B}}{\partial t^2}, \qquad c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}.$$

Τα ηλεκτρομαγνητικά κύματα διαδίδονται με σταθερή ταχύτητα c, ανεξάρτητη από την κίνηση της πηγής ή του παρατηρητή. Ωστόσο, αν εφαρμόσουμε *γαλιλαϊκή* μεταβολή συντεταγμένων στο κυματικό πρόβλημα, εμφανίζονται επιπρόσθετοι όροι (γραμμικοί και τετραγωνικοί στο V) που αλλοιώνουν τη μορφή της εξίσωσης. Έτσι, η ηλεκτρομαγνητική θεωρία δεν είναι συμβατή με τη γαλιλαϊκή συμμετρία, υπονοώντας είτε προτιμώμενο αδρανειακό πλαίσιο (αιθέρας) είτε νέα κινηματική.

3. Το πείραμα $Michelson - Morley$ και το τέλος του αιθέρα

Το πασίγνωστο συμβολόμετρο του $Michelson - Morley$ σχεδιάστηκε για να ανιχνεύσει διαφορά χρόνων διαδρομής φωτός σε κάθετο ζεύγος βραχιόνων μήκους D καθώς η Γη κινείται με ταχύτητα V ως προς τον εικαζόμενο αιθέρα. Σε κλασική ανάλυση (στο πλαίσιο του αιθέρα) οι χρόνοι είναι

$$T_{\parallel} = \frac{2Dc}{c^2 - V^2}, \qquad T_{\perp} = \frac{2D}{\sqrt{c^2 - V^2}},$$

και η διαφορά οπτικών διαδρομών προβλέπεται $\Delta L \simeq D\,V^2/c^2$, με μετρήσιμη μετατόπιση κροσσσών $\Delta n \simeq 2\Delta L/\lambda$. Παρά την υψηλή ευαισθησία, η καταγραφή ήταν *μηδενική*: καμία συστηματική μετατόπιση κατά την περιστροφή της διάταξης. Η σειρά αυτών των αποτυχιών σε διαφορετικές εποχές, τόπους και μήκη κύματος κατέστησε την υπόθεση ‘‘αιθέρα’’ περιττή: δεν υπάρχει προνομιούχο αδρανειακό πλαίσιο στην ηλεκτρομαγνητική φυσική.

4. Η ταχύτητα και η αδρανειακή μάζα του ηλεκτρονίου

Τα πειράματα του $Kaufmann$ σε δέσμες β-ηλεκτρονίων (σε εγκάρσια ηλεκτρικά και μαγνητικά πεδία) κατέγραψαν εκτροπές (x, y) πάνω σε φωτογραφικές πλάκες. Η κλασική (νευτώνεια) πρόβλεψη δίνει παραβολή $y \propto x^2$ με *σταθερή* μάζα m. Τα δεδομένα, όμως, ταίριαζαν μόνο αν η αποτελεσματική μάζα αυξάνει με την ταχύτητα. Διατυπώθηκαν ηλεκτρομαγνητικά μοντέλα (*Abraham*, *Lorentz*, *Bucherer*). Βελτιωμένες μετρήσεις $(Bucherer, Guye - -Lavanchy)$ επιβεβαίωσαν τελικά τον νόμο *Lorentz*–Άινστάιν

$$m(v) = \gamma(v)\,m_0, \qquad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}},$$

αποκλείοντας τις εναλλακτικές. Στη σχετικιστική γλώσσα, η ορμή τεσσάρων συνιστωσών αντικαθιστά την έννοια της ‘‘μεταβλητής μάζας’’ και εξηγεί γιατί κανένα υλικό σωμάτιο δεν φθάνει c.

5. Απαιτήσεις συμμετρίας και ο μετασχηματισμός *Lorentz*

Μεταξύ 1892–1904 ο *Lorentz* αναζήτησε γραμμικό μετασχηματισμό μεταξύ δύο αδρανειακών πλαισίων ώστε τα *ομογενή Maxwell* και οι κυματικές εξισώσεις να είναι μορφοαναλλοίωτα. Υπό την απαίτηση ευθύγραμμης ομαλής κίνησης $\Rightarrow$ ευθύγραμμης ομαλής κίνησης (γραμμικότητα), ισοδυναμίας των εγκάρσιων διευθύνσεων και σταθερής c, προκύπτει ο *πρότυπος μετασχηματισμός Lorentz*

$$x' = \gamma\,(x - Vt), \quad y' = y, \quad z' = z, \quad t' = \gamma\left(t - \frac{V}{c^2}\,x\right), \qquad \gamma = \frac{1}{\sqrt{1 - V^2/c^2}}.$$

Ο Ποινςαρέ ολοκλήρωσε τους κανόνες μετασχηματισμού για φορτία/ρεύματα και έδειξε ότι οι μετασχηματισμοί σχηματίζουν ομάδα. Στη σύγχρονη γλώσσα, το $F_{\mu\nu}$ και το J^μ μετασχηματίζονται ως τανυστές/τετρα-διανύσματα, ενώ η ισοδυναμία όλων των αδρανειακών πλαισίων κωδικοποιείται από την ισομετρία της Μινκόφσκι γεωμετρίας.

6. Η διατύπωση της Ειδικής Σχετικότητας

Ο Άινστάιν (1905) ανήγαγε τις παραπάνω ενδείξεις σε *δύο αξιώματα*:

1. **Αρχή της σχετικότητας:** οι νόμοι της φυσικής έχουν την ίδια μορφή σε όλα τα αδρανειακά πλαίσια.

2. **Σταθερότητα της ταχύτητας του φωτός:** η ταχύτητα του φωτός στο κενό έχει την ίδια τιμή c για κάθε αδρανειακό παρατηρητή, ανεξάρτητα από πηγή/δέκτη.

Αντί να προσαρμόσει τον *Maxwell* στη γαλιλαϊκή κινηματική ή να επαναφέρει αιθέρα, αναδιατύπωσε τον χώρο και τον χρόνο. Από τα αξιώματα αυτά προκύπτουν οι μετασχηματισμοί *Lorentz*, η αναλλοιωτική φωτεινή κωνική επιφάνεια και η ενιαία δυναμική τεσσάρων διανυσμάτων.

7. Βαθμονόμηση συστήματος αναφοράς και συγχρονισμός ρολογιών

Η σχετικιστική περιγραφή αφορά *γεγονότα* (x, y, z, t). Για λειτουργικό πλαίσιο χρειάζονται κοινοί κανόνες μέτρησης μήκους και χρόνου και ένα *σύνολο συγχρονισμένων ρολογιών* στο χώρο. Με βάση τη σταθερότητα του c: εκπέμπουμε παλμό από ρολόι A προς B, μετράμε χρόνους (t_1, t_2) στο A και t_B στο B κατά την άφιξη· ορίζουμε συγχρονισμό όταν $t_B = (t_1 + t_2)/2$ και απόσταση $D_{AB} = c\,(t_2 - t_1)/2$. Έτσι προκύπτει εσωτερικά συνεπής βαθμονόμηση ενός αδρανειακού πλαισίου. Διαφορετικά πλαίσια *δεν* συμφωνούν γενικά στη συγχρονικότητα.

8. Η σχετικότητα της ταυτόχρονης

Δύο παλμοί από τα άκρα ενός τρένου φθάνουν ταυτόχρονα στον κεντρικό παρατηρητή O' επί του τρένου, άρα (στο δικό του πλαίσιο) εκπέμφθηκαν ταυτόχρονα. Ο επί της αποβάθρας O, λόγω κίνησης του τρένου, έχει διαφορετικές αποστάσεις εκπομπής για τους δύο παλμούς, που ταξιδεύουν *με την ίδια* ταχύτητα c προς αυτόν. Για να φθάσουν ταυτόχρονα, ο παλμός από την οπίσθια άκρη πρέπει να είχε εκπέμψει νωρίτερα. Άρα γεγονότα ταυτόχρονα σε ένα πλαίσιο δεν είναι κατ΄ ανάγκη ταυτόχρονα σε άλλο: η έννοια της ''ταυτόχρονης'' είναι σχετική.

9. Παράγωγα αποτελέσματα: συστολή μήκους, διαστολή χρόνου, σύνθεση ταχυτήτων

Η μέτρηση μήκους απαιτεί *ταυτόχρονη* καταγραφή των άκρων στο ίδιο πλαίσιο. Αν L_0 είναι το ίδιομορφο (ιδιο) μήκος ράβδου εν ηρεμία και L' το μήκος της όταν κινείται με V ως προς τον μετρητή, τότε

$$L' = \frac{L_0}{\gamma} = L_0\sqrt{1 - \frac{V^2}{c^2}},$$

η γνωστή *συστολή μήκους*. Η μέτρηση χρόνου: το ιδιοχρονικό διάστημα τ μεταξύ δύο γεγονότων στο ίδιο σημείο του S παρουσιάζεται σε κινούμενο πλαίσιο ως

$$T' = \gamma\,\tau,$$

διαστολή χρόνου: ‘‘το κινούμενο ρολόι καθυστερεί’’. Η σύνθεση ταχυτήτων για κίνηση κατά x:

$$u_x = \frac{u'_x + V}{1 + \dfrac{V\,u'_x}{c^2}}, \qquad u_{y,z} = \frac{u'_{y,z}}{\gamma\left(1 + \dfrac{V\,u'_x}{c^2}\right)}.$$

Στο όριο $V \ll c$ ανακτάται ο κανόνας του Γαλιλαίου, ενώ αν $|u'_x| < c$ και $|V| < c$ τότε $|u_x| < c$ επίσης: η σύνθεση υποφωτεινών ταχυτήτων παραμένει υποφωτεινή.

1.2 Ακριβείς λύσεις των εξισώσεων Άινστάιν

1.3 Η κλάση $Weyl$ των στατικών αξονοσυμμετρικών λύσεων

Λίγο μετά τη διατύπωση των εξισώσεων πεδίου του Άινστάιν, οι $Weyl$ (1917) και $Levi\Gamma Civita$ (1918) απομόνωσαν τον στατικό, αξονοσυμμετρικό κενό τομέα και έδειξαν ότι περιγράφεται από ένα και μόνο αρμονικό δυναμικό. Σε αυτή την ενότητα επανεξετάζουμε τις *μετρικές* $Weyl$, δίνοντας έμφαση στο ότι η φαινομενική απλότητα της συμμετρίας συχνά συγκαλύπτει λεπτές ολισθήσεις στη σφαιρική δομή και στις ανωμαλίες. Δύο χαρακτηριστικά ζητήματα είναι: (ι) η σωστή ταύτιση του άξονα συμμετρίας όταν η γωνία έχει περίοδο 2π (διαφορετικά μπορεί να προκύψει ψευδής επίπεδη συμμετρία), και (ιι) η κατευθυντική φύση πιθανών ανωμαλιών στις φυσικές συντεταγμένες Ωεψλ: η προσέγγιση ενός υποτιθέμενου «ιδιοσύνολου» από διαφορετικές κατευθύνσεις στο ημιεπίπεδο (ρ, z) μπορεί να δώσει μη ισοδύναμα όρια.

Κανονική μορφή και εξισώσεις πεδίου

Κάθε στατική, αξονοσυμμετρική μετρική με δύο μεταθετικά πεδία *Killing* ∂_t και ∂_ϕ γράφεται στη μορφή *Weyl Papapetrou*

$$\mathrm{d}s^2 = -e^{2U(\eta,\xi)}\,\mathrm{d}t^2 + e^{-2U(\eta,\xi)}\big[e^{2\gamma(\eta,\xi)}\big(\mathrm{d}\eta^2 + \mathrm{d}\xi^2\big) + \rho(\eta,\xi)^2\,\mathrm{d}\phi^2\big]\,, \tag{1.1}$$

όπου U, γ, ρ εξαρτώνται μόνο από (η,ξ). Στο κενό $(\Lambda = 0)$ οι εξισώσεις Άινστάιν συνεπάγονται ότι το U ικανοποιεί εξίσωση *Laplace* σε μια *βοηθητική* επίπεδη τρισδιάστατη γεωμετρία με κυλινδρικές συντεταγμένες (ρ,ϕ,z). Στις συντεταγμένες αυτές η μετρική γράφεται

$$\mathrm{d}s^2 = -e^{2U(\rho,z)}\,\mathrm{d}t^2 + e^{-2U(\rho,z)}\big[e^{2\gamma(\rho,z)}\big(\mathrm{d}\rho^2 + \mathrm{d}z^2\big) + \rho^2\,\mathrm{d}\phi^2\big]\,, \tag{1.2}$$

με U αρμονική:

$$U_{,\rho\rho} + \frac{1}{\rho}U_{,\rho} + U_{,zz} = 0. \tag{1.3}$$

Η γραμμικότητα της (1.3) επιτρέπει υπέρθεση λύσεων. Μόλις δοθεί το U, η γ προκύπτει από ολοκληρώματα πρώτης τάξης ('τετραγωνοποιήσεις'):

$$\gamma_{,\rho} = \rho\big(U_{,\rho}^2 + U_{,z}^2\big)\,, \qquad \gamma_{,z} = 2\rho\,U_{,\rho}U_{,z}, \tag{1.4}$$

των οποίων η συμβιβαστότητα εξασφαλίζεται από την (1.3).

Για να είναι ομαλός ο άξονας συμμετρίας απαιτείται να μην υπάρχει κωνική ανωμαλία· στις συντεταγμένες *Weyl* αυτό ισοδυναμεί με $\gamma \to 0$ όταν $\rho \to 0$. Αν η συνθήκη αποτύχει, τοποθετείται έλλειμμα (ή περίσσευμα) γωνίας — μία «σπειροειδής» ανωμαλία στήριξης με τάσεις στον άξονα.

1.3.1 Επίπεδες χωροχρονικές γεωμετρίες ως μετρικές *Weyl*

Ορισμένες αρμονικές επιλογές του U αναπαριστούν τον χωροχρόνο *Minkowski* σε μη τετριμμένη χάραξη.

(ι) Τετριμμένη *Minkowski*.

$$U = 0, \qquad \gamma = 0 \quad \Longrightarrow \quad \mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}\rho^2 + \mathrm{d}z^2 + \rho^2\,\mathrm{d}\phi^2. \tag{1.5}$$

(ιι) Ομοιόμορφα επιταχυνόμενο σύστημα (*Rindler*).

$$U = \ln\rho, \qquad \gamma = \ln\rho \quad \Longrightarrow \quad \mathrm{d}s^2 = -\rho^2\,\mathrm{d}t^2 + \mathrm{d}\rho^2 + \mathrm{d}z^2 + \rho^2\,\mathrm{d}\phi^2, \tag{1.6}$$

που είναι εκ νέου επίπεδη, αλλά σε συντεταγμένες τύπου *Rindler*, με την επιφάνεια $\rho = 0$ να λειτουργεί ως ορίζοντας επιτάχυνσης (με κατάλληλη μεταβολή μεταβλητών).

(ιιι) Πλαίσιο *Gautreau Hoffman*.

$$U = \tfrac{1}{2}\ln\bigl(\sqrt{\rho^2 + z^2} + z\bigr), \qquad \gamma = \tfrac{1}{2}\ln\frac{\sqrt{\rho^2 + z^2} + z}{2\sqrt{\rho^2 + z^2}}, \tag{1.7}$$

παράγει επίσης μηδενική καμπυλότητα. Αν και θυμίζει ημι-άπειρη Νευτώνεια ράβδο με γραμμική πυκνότητα $\sigma = \frac{1}{2}$ στον αρνητικό άξονα z, μια ολική αλλαγή συντεταγμένων αποκαλύπτει ξανά επίπεδο χωροχρόνο, ιδωμένο από ομοιόμορφα επιταχυνόμενους παρατηρητές, με το «σώμα» πάνω σε ορίζοντα. Το δίδαγμα είναι σαφές: το βοηθητικό δυναμικό U δεν εγγυάται μόνο του πραγματική κατανομή μάζας στη 4-διάστατη γεωμετρία.

1.3.2 Πολυπολική ανάπτυξη και οικογένεια *Weyl*

Εισάγουμε σφαιρικές μεταβλητές στον βοηθητικό χώρο,

$$\rho = r\sin\theta, \qquad z = r\cos\theta, \tag{1.8}$$

ώστε η (1.3) να γίνεται η αξονοσυμμετρική *Laplace*:

$$r^2 U_{,rr} + 2rU_{,r} + U_{,\theta\theta} + \cot\theta\, U_{,\theta} = 0. \tag{1.9}$$

Οι ασυμπτωτικά επίπεδες, κανονικές λύσεις παραδέχονται ανάπτυξη σε πολυώνυμα *Legendre*:

$$U(r,\theta) = -\sum_{n=0}^{\infty} a_n\, r^{-(n+1)} P_n(\cos\theta), \tag{1.10}$$

όπου οι συντελεστές $\{a_n\}$ παίζουν τον ρόλο «μαζικών πολυπολικών παραμέτρων». Η γ γράφεται ως διπλή σειρά,

$$\gamma(r,\theta) = -\sum_{l,m=0}^{\infty} a_l a_m\, \frac{(l+1)(m+1)}{l+m+2}\, \frac{P_l(\cos\theta)P_m(\cos\theta) - P_{l+1}(\cos\theta)P_{m+1}(\cos\theta)}{r^{\,l+m+2}}, \tag{1.11}$$

η οποία κωδικοποιεί τη μη γραμμική αυτο-αλληλεπίδραση των Νευτώνειων πολυπόλων. Η κανονικότητα στον άξονα και η καλή ασυμπτωτική συμπεριφορά επιβάλλουν επιπλέον περιορισμούς στις επιτρεπτές ακολουθίες $\{a_n\}$.

Μετρική *Levi Civita* (άπειρη ράβδος). Μία χαρακτηριστική υποοικογένεια γράφεται

$$\mathrm{d}s^2 = -\rho^{4\sigma}\,\mathrm{d}t^2 + k^2\,\rho^{4\sigma(2\sigma-1)}\big(\mathrm{d}\rho^2 + \mathrm{d}z^2\big) + \rho^{2(1-2\sigma)}\,\mathrm{d}\phi^2, \tag{1.12}$$

που αντιστοιχεί σε

$$U(\rho) = 2\sigma\ln\rho, \qquad \gamma(\rho) = 4\sigma^2\ln\rho + \ln k. \tag{1.13}$$

Εδώ το U ταυτίζεται με το Νευτώνειο δυναμικό άπειρης ράβδου γραμμικής πυκνότητας σ. Γενικά εμφανίζεται κωνική ανωμαλία στον άξονα ($\rho = 0$), με έλλειμμα γωνίας ανάλογο της σ· για μεγάλες τιμές σ προκύπτουν πρόσθετες παθολογίες (π.χ. μεγάλοι ερυθρομετατοπισμοί και τάσεις). Το παράδειγμα υπογραμμίζει ότι απλές Νευτώνειες αναλογίες συχνά αποκρύπτουν μη τετριμμένη 4-διάστατη γεωμετρία.

1.3.3 *Schwarzschild* σε συντεταγμένες *Weyl*

Η μοναδική στατική, σφαιρικά συμμετρική κενή λύση μπορεί να μετασχηματιστεί σε μορφή *Weyl* μέσω επίμηκων σφαιροειδών συντεταγμένων:

$$r = m(x+1), \qquad y = \cos\theta, \qquad \rho = m\sqrt{(x^2-1)(1-y^2)}, \qquad z = mxy, \tag{1.14}$$

με $x > 1$ να περιγράφει τη στατική περιοχή $r > 2m$. Στις μεταβλητές αυτές

$$e^{2U} = \frac{R_+ + R_- - 2m}{R_+ + R_- + 2m}, \qquad e^{2\gamma} = \frac{(R_+ + R_-)^2 - 4m^2}{4R_+R_-}, \qquad R_\pm = \sqrt{\rho^2 + (z \pm m)^2}, \tag{1.15}$$

ή ισοδύναμα

$$U = \tfrac{1}{2}\ln\Big(\frac{R_- + z - m}{R_+ + z + m}\Big). \tag{1.16}$$

Το U συμπίπτει με το δυναμικό πεπερασμένης Νευτώνειας ράβδου μήκους $2m$ και πυκνότητας $\sigma = \frac{1}{2}$ στο διάστημα $-m < z < m$. Η αντιστοίχιση τμημάτων του άξονα *Weyl* με γνωστές περιοχές *Schwarzschild* είναι:

Περιοχή *Schwarzschild*	$\leftrightarrow$	Τμήμα άξονα *Weyl*
$\theta = \pi$, $r \geq 2m$	$\leftrightarrow$	$\rho = 0,\ z \leq -m,$
$0 < \theta < \pi$, $r = 2m$ (ορίζοντας)	$\leftrightarrow$	$\rho = 0,\ -m < z < m,$
$\theta = 0$, $r \geq 2m$	$\leftrightarrow$	$\rho = 0,\ z \geq m.$

Παρότι το σχήμα της ράβδου είναι εύγλωττο, χρειάζεται προσοχή: η «ράβδος» αναπαριστά τον ορίζοντα Κιλλινγ και όχι υλικό σώμα. Διασπάσεις σε ημι-άπειρες ράβδους αντίθετης πυκνότητας αναπαράγουν σωστά το U, αλλά θολώνουν τη σωστή ολική γεωμετρία.

1.3.4 *Zipoy Voorhees*

Μία φυσική μονοπαραμετρική παραμόρφωση του *Schwarzschild* μέσα στην κλάση *Weyl* προκύπτει αν αναθέσουμε αυθαίρετη σταθερή πυκνότητα σ σε πεπερασμένη ράβδο ημι-μήκους ℓ:

$$U(\rho, z) = \sigma \ln \frac{R_- + z - \ell}{R_+ + z + \ell}, \qquad R_\pm = \sqrt{\rho^2 + (z \pm \ell)^2}. \tag{1.17}$$

Θέτοντας ολική μάζα $m = 2\sigma\ell$ και $\delta \equiv m/\ell$, παίρνουμε

$$e^{2U} = \left(\frac{R_+ + R_- - 2\ell}{R_+ + R_- + 2\ell} \right)^\delta, \qquad e^{2\gamma} = \left(\frac{(R_+ + R_-)^2 - 4\ell^2}{4R_+R_-} \right)^{\delta^2}, \tag{1.18}$$

τη μετρική *Zipoy Voorhees* (ή *γ-μετρική*). Ειδικά όρια: $\delta = 1$ (*Schwarzschild*), $\ell \to 0$ με m σταθερό (*Curzon Chazy* σημειακή μάζα), και $\ell \to \infty$ με $m/(2\ell) \to 0$ (*Levi Civita* άπειρη γραμμή). Εκτός από $\delta = 0, 1$, το τμήμα $\rho = 0$, $|z| < \ell$ είναι γυμνή καμπυλωτική ανωμαλία. Η περιφέρεια μικρού κύκλου γύρω από τον άξονα συμπεριφέρεται ως

$$C(\rho) \approx 2\pi\, \rho^{1-\delta}\, \ell^\delta \qquad (\rho \ll \ell),$$

που εκρήγνυται για $\delta > 1$ και μηδενίζεται για $0 < \delta < 1$, δηλώνοντας έντονη κατευθυντική ανισοτροπία κοντά στον άξονα.

Σε επίμηκες σφαιροειδές σύστημα,

$$\rho = \ell\sqrt{(x^2 - 1)(1 - y^2)}, \qquad z = \ell xy,$$

η μετρική απλοποιείται σε

$$\mathrm{d}s^2 = -\left(\frac{x-1}{x+1}\right)^{\delta} \mathrm{d}t^2 + \Sigma^2\left(\frac{\mathrm{d}x^2}{x^2-1} + \frac{\mathrm{d}y^2}{1-y^2}\right) + R^2\,\mathrm{d}\phi^2, \tag{1.19}$$

με

$$e^{-2\gamma} = \left(\frac{x^2-1}{x^2-y^2}\right)^{\delta^2}, \tag{1.20}$$

$$\Sigma^2 = \ell^2(x+1)^{\delta+1}(x-1)^{\delta-1}(x^2-y^2)^{1-\delta^2}, \tag{1.21}$$

$$R^2 = \ell^2(x+1)^{1+\delta}(x-1)^{1-\delta}(1-y^2). \tag{1.22}$$

Για $\delta \neq 0, 1$ ο τόπος $x = 1$ ($\rho = 0$, $|y| < 1$) αποτελεί γνήσια, κατευθυντικά εξαρτώμενη ανωμαλία και όχι κανονικό ορίζοντα.

1.4 Ο μηδενισμός των μεγεθών Λοβ για μελανές οπές τύπου Κερρ και ο ρόλος των συμμετριών

Σημείωση: Το ακόλουθο κεφάλαιο έχει γραφτεί από εμένα, τον επιβλέποντά μου Α. Κεηαγιας, καθώς και τον συνάδελφο του επιβλέποντά μου Α. Ριοττο, και έχει δημοσιευθεί στο $JCAP$.

1.5 Εισαγωγή

Τα Βαρυτικά Κύματα (GWs) και οι Μαύρες Τρύπες (BHs) αποτελούν κεντρικές προβλέψεις της Γενικής Σχετικότητας (GR), οι οποίες έχουν επιβεβαιωθεί πειραματικά από πρωτοποριακές παρατηρήσεις, όπως η ανίχνευση GWs από συγχωνεύσεις BHs από τις συνεργασίες ΛΙΓΟ και ΐργο [1]. Οι ανιχνεύσεις αυτές παρέχουν κρίσιμες ενδείξεις υπέρ της θεωρίας βαρύτητας του Άινστάιν, χωρίς να δείχνουν αποκλίσεις από αυτήν [35].

Κατά τη φάση σπειροειδούς έλικας (ινσπιραλ) ενός συμπαγούς δυαδικού συστήματος, όπως εκείνων που περιλαμβάνουν αστέρες νετρονίων ή BHs, οι παλιρροϊκές αλληλεπιδράσεις καθίστανται σημαντικές όταν η τροχιακή απόσταση γίνει επαρκώς μικρή. Τα παλιρροϊκά αυτά φαινόμενα επηρεάζουν τόσο τη δυναμική του συστήματος όσο και τα εκπεμπόμενα GWs. Η σύζευξη GWs–παλιρροϊκών επιδράσεων είναι ουσιώδης για την τελειοποίηση των μοντέλων ινσπιραλ και για τον έλεγχο της GR σε ακραίες συνθήκες.

Οι παλιρροϊκές επιδράσεις ποσοτικοποιούνται από παραμέτρους γνωστές ως Αριθμοί *Love* (*Love numbers*), οι οποίοι μετρούν τη διαμόρφωση ενός αντικειμένου υπό την επίδραση του βαρυτικού πεδίου του συνοδού του. Ειδικότερα, οι στατικοί Παλιρροϊκοί Αριθμοί *Love* (TLN) εξαρτώνται από την εσωτερική δομή και σύσταση των συμπαγών αντικειμένων που υφίστανται παλιρροϊκή παραμόρφωση [42]. Οι παράμετροι αυτοί παίζουν καίριο ρόλο στην τροποποίηση του βαρυτικού κυματομορφικού σήματος, με τις συνεισφορές τους να εμφανίζονται στην πέμπτη μετα-Νευτώνεια τάξη [20]. Για παράδειγμα, οι μη μηδενικοί TLN των αστέρων νετρονίων παρέχουν πολύτιμες πληροφορίες για την εξίσωση κατάστασης της πυκνής πυρηνικής ύλης. Αντιθέτως, για τις BHs αναμένεται μηδενικός TLN λόγω απουσίας άκαμπτης δομής. Αυτό συνήθως αποδεικνύεται με θεωρία διαταραχών, δείχνοντας ότι μια γραμμική παλιρροϊκή παραμόρφωση με πλάτος ανάλογο του r^{ℓ} δεν προκαλεί απόκριση $r^{-\ell-1}$ (με ℓ τον αντίστοιχο πολυπολικό δείκτη), οδηγώντας σε μηδενικούς στατικούς TLN. Γραμμικές διαταραχές προκαλούμενες από εξωτερικές παλιρροϊκές δυνάμεις δεν μπορούν να παραγάγουν μη μηδενικούς TLN [5, 16, 15, 39, 38, 43, 36, 12, 37, 41, 33]. Το φαινόμενο αυτό φαίνεται να ανάγεται σε υποκείμενες κρυφές συμμετρίες [25, 10, 9, 26, 27, 11, 28, 30, 6, 32, 3, 4, 17, 45].

Πρόσφατες αναλύσεις επιβεβαίωσαν ότι οι στατικοί TLN μηδενίζονται και για διαταραχές δεύτερης τάξης στο εξωτερικό παλιρροϊκό πεδίο [47, 46]. Επιπλέον, για τη *Schwarzschild BH* έχει αποδειχθεί ότι η μηδενικότητα των TLN ισχύει για τις άρτιας παράτητας (παριτψ-εεν) διαταραχές σε όλες τις τάξεις στο εξωτερικό παλιρροϊκό πεδίο [34, 13].

Το αν ο στατικός TLN των BHs μηδενίζεται ή όχι είναι κεφαλαιώδους σημασίας για τη διάκριση συγχωνεύσεων BH BH από συγχωνεύσεις αστέρων νετρονίων [14], καθώς οι αστέρες νετρονίων διαθέτουν σημαντικό TLN. Περαιτέρω, ακόμη και η συγχώνευση δύο μη στροφικών BHs παράγει τελικώς μια στροφική $Kerr$ BH. Αυτό θέτει το αναπόδραστο ερώτημα: έχουν οι $Kerr$ BHs μηδενικό στατικό TLN σε κάθε τάξη του εξωτερικού παλιρροϊκού πεδίου·

Η περίπτωση των στροφικών BHs, που περιγράφονται από τη λύση $Kerr$, παρουσιάζει πρόσθετες δυσκολίες. Η περιστροφή εισάγει σύρση αδρανειακών συστημάτων ($frame$ $dragging$) και τροποποιεί τη γεωμετρία του χωροχρόνου, περιπλέκοντας την ανάλυση της παλιρροϊκής απόκρισης. Η κατανόηση της παλιρροϊκής απόκρισης των $Kerr$ BHs είναι ουσιώδης, όχι μόνο για θεωρητική πληρότητα, αλλά και για τη μοντελοποίηση κυματομορφών GWs σε ρεαλιστικά αστροφυσικά συστήματα, όπου οι BHs αναμένονται συχνά να περιστρέφονται.

Σε αυτή την εργασία αντιμετωπίζουμε το ερώτημα της μηδενικότητας των στατικών TLN των $Kerr$ BHs χρησιμοποιώντας το φορμαλισμό $Ernst$ [18] και τις συντεταγμένες $Weyl$ για την ανάλυση της παλιρροϊκής απόκρισης. Το δυναμικό $Ernst$ προσφέρει ένα ισχυρό πλαίσιο περιγραφής αξονοσυμμετρικών χωροχρονικών, επιτρέποντάς μας να ενσωματώσουμε συστηματικά την περιστροφή και τις μη γραμμικότητες. Εκφράζοντας τη μετρική $Kerr$ σε επιμήκεις σφαιροειδείς συντεταγμένες, γενικεύουμε προηγούμενα αποτελέσματα για $Schwarzschild$ BHs και αποδεικνύουμε ότι οι στατικοί παλιρροϊκοί αριθμοί $Love$ των $Kerr$ BHs μηδενίζονται σε όλες τις τάξεις του εξωτερικού παλιρροϊκού πεδίου. Θα εντοπίσουμε επίσης τις μη γραμμικές συμμετρίες που ευθύνονται για αυτό το αποτέλεσμα.

Το εύρημα αυτό υπογραμμίζει τη στιβαρότητα των επιχειρημάτων βασισμένων σε συμμετρίες που διέπουν τις αποκρίσεις των BHs και αναδεικνύει τη διακριτή φύση των BHs ως λύσεων της GR. Η μηδενικότητα των TLN επιβεβαιώνει την αρχή ότι οι BHs, σε αντίθεση με άλλα συμπαγή αντικείμενα, δεν διατηρούν καμία μόνιμη παραμόρφωση υπό στατικές παλιρροϊκές δυνάμεις. Η μελέτη αυτή συμβάλλει στην ευρύτερη κατανόηση της φυσικής των BHs, προσφέροντας νέες οπτικές για την αλληλεπίδρασή τους με εξωτερικά πεδία και τις επιπτώσεις στην αστρονομία βαρυτικών κυμάτων.

Πρέπει, ωστόσο, να τονίσουμε ότι, σε αντίθεση με τη μάζα και το σπιν των BHs, που είναι καλώς ορισμένα διατηρούμενα φορτία και βαθμονομητικά αμετάβλητες ποσότητες, οι παλιρροϊκοί αριθμοί $Love$ έχουν διαφορετική φύση. Ενώ στον Νευτώνειο ορισμό τους είναι άμεσοι [24], ο στατικός TLN δεν αποτελεί διατηρούμενο φορτίο ούτε μια καθολικά βαθμονομητικά αμετάβλητη ποσότητα στη Γενική Σχετικότητα. Αυτό έχει συζητηθεί εκτενώς στη βιβλιογραφία (βλ. π.χ. [23, 5, 29, 31]).

Η ασάφεια αυτή οδήγησε σε μια εναλλακτική προσέγγιση: τον ορισμό του γραμμικού στατικού TLN ως συντελεστή $Wilson$ που προκύπτει από αντιστοίχιση ($matching$) ενός τελεστή στη δράση $worldline$. Ωστόσο, η αντιστοίχιση αυτή προϋποθέτει μια συγκεκριμένη επιλογή βαθμίδας—τυπικά τη βαθμίδα $deDonder$—για την απλοποίηση του διαδότη του γκραβιτόνιου. Κατά συνέπεια, είναι αναγκαία η μετάφραση των αποτελεσμάτων από τη βαθμίδα $deDonder$ σε άλλη βαθμίδα, όπως η βαθμίδα $Regge\Gamma Wheeler(RW)$ που χρησιμοποιείται στο γραμμικό επίπεδο, για την ολοκλήρωση της διαδικασίας αντιστοίχισης.

Μια φυσική επιθυμία είναι να διατυπωθεί ένας βαθμονομητικά αμετάβλητος ορισμός του TLN. Αυτό, όμως, δεν είναι ευθύγραμμο, καθώς μπορούν να οικοδομηθούν άπειρες βαθμονομητικά αμετάβλητες ποσότητες μόλις επιλεγεί μια βαθμιδο-δεσμευμένη έκφραση. Μια εναλλακτική στρατηγική είναι η εργασία σε κατάλληλη βαθμίδα. Η βέλτιστη βαθμίδα εξαρτάται από το συγκεκριμένο πλαίσιο μέτρησης. Για παράδειγμα, στην κοσμολογία, η παράμετρος $halobias$ ορίζεται φυσικότερα σε συγχρονικές συντεταγμένες, οι οποίες χρησιμοποιούνται στο μοντέλο σφαιρικής κατάρρευσης [50]. Η πρόκληση με τον στατικό TLN είναι ότι δεν μετριέται απευθείας αλλά συναγάγεται μέσω $Bayesian$ ανάλυσης βασισμένης σε προσαρμογή κυματομορφών που εξαρτάται από το μοντέλο.

Από την άλλη, ένα σημαντικό πλεονέκτημα της μεθόδου μας είναι ο μη διαταρακτικός της χαρακτήρας, καθώς βρίσκουμε ακριβή λύση των πλήρων εξισώσεων Άινστάιν για έναν στατικό και αξονοσυμμετρικό χωροχρόνο κενού. Η μηδενικότητα του *Love number* προκύπτει κατόπιν ανάλυσης της συμπεριφοράς του βαθμωτού *Kretschmann* κοντά στον ορίζοντα γεγονότων. Δεδομένου ότι το *Kretschmann* κωδικοποιεί εγγενείς ιδιότητες καμπυλότητας, το συμπέρασμα αυτό είναι τελικώς ανεξάρτητο της επιλογής συντεταγμένων.

Η εργασία οργανώνεται ως εξής: Η Ενότητα 2 ανασκοπεί την κλάση *Weyl* των στατικών, αξονοσυμμετρικών λύσεων κενού και εισάγει τον φορμαλισμό του δυναμικού *Ernst*. Η Ενότητα 3 επανεξετάζει την παλιρροϊκή απόκριση των *Schwarzschild BHs*, θέτοντας το πλαίσιο για μη γραμμικά παλιρροϊκά φαινόμενα. Η Ενότητα 4 επεκτείνει την ανάλυση στις *Kerr BHs*, περιγράφοντας τη μετάβαση σε επιμήκεις σφαιροειδείς συντεταγμένες και εξετάζοντας τις φθίνουσες και αυξανόμενες τετραπολικές ιδιομορφές. Η Ενότητα 5 μελετά την επίδραση των μη γραμμικών παλιρροϊκών αλληλεπιδράσεων και τον ρόλο τους στη διασφάλιση της μηδενικότητας των *TLN*. Η Ενότητα 6 συζητά τον ρόλο των μη γραμμικών συμμετριών. Η Ενότητα 7 κλείνει με τις επιπτώσεις και πιθανές επεκτάσεις της εργασίας. Τέλος, τα Παραρτήματα Α και Β συζητούν τη μετάβαση σε συντεταγμένες *Boyer Lindquist* και άλλες βάσεις πολυπόλων, προσφέροντας συμπληρωματική οπτική.

1.6 Η κλάση *Weyl* των στατικών, αξονοσυμμετρικών λύσεων κενού

Όπως έδειξε ο *Ernst* [18], οι εξισώσεις πεδίου για ομοιόμορφα περιστρεφόμενη, αξονοσυμμετρική πηγή μπορούν να αναδιατυπωθούν μέσω ενός απλού μεταβλητοτικού (*variational*) αρχής. Ακολουθώντας αυτή την προσέγγιση αναδύονται ενιαίες λύσεις για τις μετρικές *Weyl* και Παπαπετρου, οι οποίες μας δίνουν άμεση παραγωγή τόσο της

Schwarzschild όσο και της *Kerr* μετρικής σε επιμήκεις σφαιροειδείς συντεταγμένες. Νέες λύσεις για την *Kerr BH* σε παλιρροϊκά περιβάλλοντα μπορούν επίσης να ληφθούν με αυτό τον τρόπο, επιτρέποντάς μας δηλώσεις για τους μη γραμμικούς στατικούς *Love number*ς των *Kerr BHs*. Ξεκινάμε εξετάζοντας μια στατική, αξονοσυμμετρική μετρική *Weyl* στη μορφή [40]

$$\mathrm{d}s^2 = f^{-1}\left[e^{2\gamma}(\mathrm{d}\rho^2 + \mathrm{d}z^2) + \rho^2\mathrm{d}\varphi^2\right] - f(\mathrm{d}t - \omega\mathrm{d}\varphi)^2, \tag{1.23}$$

όπου $f = f(\rho, z)$, $\omega = \omega(\rho, z)$ και $\gamma = \gamma(\rho, z)$.[1] Προκύπτει ότι οι εξισώσεις για τα f και ω, που απορρέουν από τις εξισώσεις πεδίου κενού του Άινστάιν ($R_{\mu\nu} = 0$), αποζευγνύονται από την εξίσωση για τη $\gamma(\rho, z)$ και δίνονται από

$$f\nabla^2 f = \nabla f \cdot \nabla f - \rho^{-2} f^4 \nabla\omega \cdot \nabla\omega, \tag{1.24}$$

$$\nabla \cdot \left(\rho^{-2} f^2 \nabla\omega\right) = 0. \tag{1.25}$$

Εισάγουμε τώρα νέο βαθμωτό ϕ από την ω ως

$$\nabla\phi = -\frac{f^2}{\rho}\hat{n}_\varphi \times \nabla\omega \tag{1.26}$$

όπου $\hat{n}_\varphi$ είναι το μοναδιαίο διάνυσμα στη διεύθυνση φ. Έχει δειχθεί [18] ότι οι (6.2) και (6.3) μπορούν επίσης να εξαχθούν μέσω μιας μιγαδικής συνάρτησης, του δυναμικού *Ernst* $\mathcal{E}$, οριζόμενου ως

$$\mathcal{E} = f + i\phi. \tag{1.27}$$

[1] Σημειώστε ότι για $\rho \to 0$ θα πρέπει να ισχύει $\gamma \to 0$, διότι αλλιώς η μετρική θα περιείχε τμήμα ανάλογο με $e^{2\gamma(0,z)}\mathrm{d}\rho^2 + \rho^2\mathrm{d}\varphi^2$, το οποίο σαφώς φέρει κωνική ανωμαλία για οποιοδήποτε z.

Περαιτέρω, οι εξισώσεις για τη τρίτη συνάρτηση $\gamma(r,\theta)$ γράφονται όροι του $\mathcal{E}$ ως [19]

$$\begin{aligned} \gamma_{,z} &= \frac{1}{4}\rho f^{-2}\left[(\mathcal{E}_{,\rho})(\mathcal{E}^*_{,z}) + (\mathcal{E}_{,z})(\mathcal{E}^*_{,\rho})\right], \\ \gamma_{,\rho} &= \frac{1}{4}\rho f^{-2}\left[(\mathcal{E}_{,\rho})(\mathcal{E}^*_{,\rho}) - (\mathcal{E}_{,z})(\mathcal{E}^*_{,z})\right]. \end{aligned} \tag{1.28}$$

Εισάγουμε επιμήκεις σφαιροειδείς συντεταγμένες (t, x, y, φ) αντί των συντεταγμένων $Weyl$ γράφοντας [52, 44]

$$\begin{aligned} &\rho = \rho_0(x^2-1)^{1/2}(1-y^2)^{1/2}, \quad x \geq 1, \quad |y| \leq 1, \\ &z = \rho_0 xy, \quad \rho_0 = \text{σταθ.} \end{aligned} \tag{1.29}$$

Θα δούμε παρακάτω ότι το ρ_0 σχετίζεται με τη μάζα και την παράμετρο σπιν των BHs που θέλουμε να περιγράψουμε. Στις συντεταγμένες αυτές, η μετρική της (6.1) γράφεται

$$\mathrm{d}s^2 = \rho_0^2 f^{-1}\left[e^{2\gamma}(x^2-y^2)\left(\frac{\mathrm{d}x^2}{x^2-1} + \frac{\mathrm{d}y^2}{1-y^2}\right) + (x^2-1)(1-y^2)\mathrm{d}\varphi^2\right] - f(\mathrm{d}t - \omega \mathrm{d}\varphi)^2. \tag{1.30}$$

Για μελλοντική χρήση, οι διαφορικοί τελεστές που εισήχθησαν προηγουμένως, στις επιμήκεις σφαιροειδείς συντεταγμένες παίρνουν τη μορφή:

$$\begin{aligned} \nabla &\equiv \rho_0^{-1}(x^2-y^2)^{-1/2}\left[\hat{n}_x(x^2-1)^{1/2}\partial_x + \hat{n}_y(1-y^2)^{1/2}\partial_y\right], \\ \nabla^2 &= \rho_0^{-2}(x^2-y^2)^{-1}\left\{\partial_x\Big[(x^2-1)\partial_x\Big] + \partial_y\Big[(1-y^2)\partial_y\Big]\right\}, \end{aligned} \tag{1.31}$$

ενώ το εσωτερικό γινόμενο των βαθμίδων δύο συναρτήσεων A και B είναι

$$\nabla A \cdot \nabla B = \rho_0^{-2}(x^2-y^2)^{-1}\Big[(x^2-1)\partial_x A\, \partial_x B + (1-y^2)\partial_y A\, \partial_y B\Big].$$

Οι εξισώσεις (6.2) και (6.3) ισοδυναμούν με την εξίσωση κίνησης για το δυναμικό $Ernst$

$\mathcal{E}$ που προκύπτει από τη δράση

$$\mathcal{S}_{\mathcal{E}} = \int \frac{\nabla\mathcal{E}\cdot\nabla\mathcal{E}^*}{(\mathcal{E}+\mathcal{E}^*)^2} d^2x, \tag{1.32}$$

έτσι ώστε οι αντίστοιχες εξισώσεις

$$(\mathcal{E}+\mathcal{E}^*)\,\nabla^2\mathcal{E} - \nabla\mathcal{E}\cdot\nabla\mathcal{E} = 0, \tag{1.33}$$

αναπαράγουν τις (6.2) και (6.3). Κατά συνέπεια, το πρόβλημα εύρεσης αξονοσυμμετρικών, στάσιμων λύσεων κενού των εξισώσεων Άινστάιν ανάγεται σε κατάλληλη επίλυση της (6.11) για το δυναμικό *Ernst* $\mathcal{E}$.

1.7 Η *Schwarzschild BH* σε εξωτερικά παλιρροϊκά πεδία

Παρότι η *Schwarzschild BH* σε εξωτερικά παλιρροϊκά πεδία έχει συζητηθεί εκτενώς στο [34], υπενθυμίζουμε εδώ την περιγραφή της όρους του δυναμικού *Ernst*. Για τη στατική *Schwarzschild* με $\omega = 0$ και σε επιμήκεις σφαιροειδείς συντεταγμένες, το δυναμικό είναι πραγματικό και δίνεται από

$$\mathcal{E} = e^{2\psi}\frac{x-1}{x+1}, \tag{1.34}$$

όπου $\psi(x,y)$ είναι πραγματικό δυναμικό. Υποκαθιστώντας την παραπάνω έκφραση στην εξίσωση κίνησης (6.11), βρίσκουμε ότι η ψ ικανοποιεί την εξίσωση *Laplace*

$$\nabla^2\psi = 0. \tag{1.35}$$

Τονίζουμε ότι, μολονότι η (6.11) είναι μη γραμμική εξίσωση που ενσωματώνει τη μη γραμμικότητα των εξισώσεων Άινστάιν, η (6.13) είναι ακριβής *γραμμική* εξίσωση που ικανοποιεί η ψ. Όλη η μη γραμμικότητα μεταφέρθηκε στη συνάρτηση γ, η οποία καθορίζεται από τη μη γραμμική (6.6). Αυτή είναι η δύναμη της μεθόδου *Ernst*, όπου μια γραμμική εξίσωση απομονώνεται από το πλήρες μη γραμμικό σύστημα. Η λύση για τη $\psi(x,y)$ γράφεται ως πολυπολική ανάπτυξη

$$\psi = \sum_{\ell \geq 1} U_\ell(x) Y_\ell(y) \ , \tag{1.36}$$

όπου U_ℓ και Y_ℓ ικανοποιούν

$$\frac{\mathrm{d}}{\mathrm{dx}}\left((x^2-1)\frac{\mathrm{d}}{\mathrm{dx}}U_\ell\right) - \ell(\ell+1)U_\ell = 0, \tag{1.37}$$

$$\frac{\mathrm{d}}{\mathrm{dy}}\left((1-y^2)\frac{\mathrm{d}}{\mathrm{dy}}Y_\ell\right) + \ell(\ell+1)Y_\ell = 0. \tag{1.38}$$

Η κανονική λύση της (6.16) στα $y = \pm 1$ δίνεται από τα πολυώνυμα *Legendre*

$$Y_\ell(y) = P_\ell(y), \qquad \ell = 0, 1, \cdots , \tag{1.39}$$

και, ομοίως, η λύση της (6.15) είναι

$$U_\ell = \alpha_\ell \, x^\ell \, {}_2F_1\left(\frac{1-\ell}{2}, -\frac{\ell}{2}, \frac{1-2\ell}{2}, \frac{1}{x^2}\right) + \beta_\ell \frac{1}{x^{\ell+1}} \, {}_2F_1\left(\frac{1+\ell}{2}, \frac{2+\ell}{2}, \frac{3+2\ell}{2}, \frac{1}{x^2}\right), \tag{1.40}$$

ώστε η $\psi(x,y)$ προκύπτει

$$\psi(x,y) = \sum_{\ell=1}^{\infty}\left[\alpha_\ell \, x^\ell \, {}_2F_1\left(\frac{1-\ell}{2}, -\frac{\ell}{2}, \frac{1-2\ell}{2}, \frac{1}{x^2}\right) + \frac{\beta_\ell}{x^{\ell+1}} \, {}_2F_1\left(\frac{1+\ell}{2}, \frac{2+\ell}{2}, \frac{3+2\ell}{2}, \frac{1}{x^2}\right)\right] P_\ell(y). \tag{1.41}$$

Έχει δειχθεί στο [34] ότι η φθίνουσα ιδιομορφή (ανάλογη του $r^{-\ell-1}$) γεννά γυμνή α-νωμαλία στον ορίζοντα $x = 1$.[2] Μπορεί να επαληθευθεί εξετάζοντας τον βαθμωτό *Kretschmann*, ο οποίος αποκλίνει για $\beta_\ell \neq 0$ στο $x = 1$, όπως ήδη έδειξε το [34]. Αντίθετα, η αυξανόμενη ιδιομορφή (ανάλογη του r^ℓ) δεν είναι ιδιάζουσα στον ορίζοντα. Επομένως, $\beta_\ell = 0$, οδηγώντας σε μηδενικό στατικό *Love number* για *Schwarzschild BH* σε εξωτερικό βαρυτικό πεδίο σε όλες τις τάξεις στο παλιρροϊκό παράμετρο [34].

Σημειώνουμε επίσης ότι η λύση (6.19) καθορίζει και τη συνάρτηση $\gamma(x,y) = \gamma_s(x,y)$ για τη *Schwarzschild BH* μέσω των εξισώσεων (6.6), που σε επιμήκεις σφαιροειδείς συντεταγμένες γράφονται ρητά ως

$$\begin{aligned}
\gamma_{s,x} &= \frac{1-y^2}{x^2-y^2}\Big[x\left(x^2-1\right)U_{,x}^2 - x\left(1-y^2\right)U_{,y}^2 - 2y\left(x^2-1\right)U_{,x}U_{,y}\Big],\\
\gamma_{s,y} &= \frac{x^2-1}{x^2-y^2}\Big[y\left(x^2-1\right)U_{,x}^2 - y\left(1-y^2\right)U_{,y}^2 + 2x\left(1-y^2\right)U_{,x}U_{,y}\Big],
\end{aligned} \tag{1.42}$$

όπου

$$U(x,y) = \frac{1}{2}\ln\left(\frac{x-1}{x+1}\right) + \psi(x,y). \tag{1.43}$$

Τότε, η γενική λύση για $\gamma_s(x,y)$, δίνεται από τον κλειστό τύπο [52]

$$\gamma_s(x,y) = (x^2-1)\int_{-1}^{y}\frac{\Gamma(x,y')}{x^2-y'^2}\,\mathrm{d}y', \tag{1.44}$$

όπου

$$\Gamma(x,y) = y(x^2-1)U_{,x}^2 - y(1-y^2)U_{,y}^2 + 2x(1-y^2)U_{,x}U_{,y}\ .$$

[2] Η υπερεπιφάνεια $x = 1$ είναι πράγματι ορίζοντας, καθώς η $f = \mathrm{Re}(\mathcal{E})$ στη (6.1) μηδενίζεται: $f(x=1) = 0$. Αυτό φαίνεται και από τον ορισμό των επιμήκων συντεταγμένων (6.7) που καλύπτουν την περιοχή $x \in [1,\infty)$, δηλ. την εξωτερική του ορίζοντα.

1.8 Η *Kerr BH* σε εξωτερικά παλιρροϊκά πεδία

Για να εισαγάγουμε περιστροφή, πρέπει να επιτρέψουμε μη μηδενική ω στη μετρική (6.1). Σε αυτή την περίπτωση αναμένουμε η (6.1) να περιγράψει την *Kerr BH* καθώς και την εμβάπτισή της σε εξωτερικά παλιρροϊκά πεδία, κατ' ανάλογο τρόπο με το μη περιστρεφόμενο υπόβαθρο *Schwarzschild* της προηγούμενης ενότητας. Εφόσον για περιστρεφόμενη *BH* η ω δεν μηδενίζεται, το δυναμικό *Ernst* πρέπει να έχει μη μηδενικό φανταστικό μέρος ϕ, το οποίο καθορίζεται από την (6.4).

Ειδικότερα, έχει δειχθεί [7, 48] ότι η κατάλληλη επιλογή για το δυναμικό *Ernst* μιας *Kerr BH* σε εξωτερικό παλιρροϊκό πεδίο έχει τη μορφή

$$\mathcal{E} = e^{2\psi}\, \frac{x(1+ab) + iy(b-a) - (1-ia)(1-ib)}{x(1+ab) + iy(b-a) + (1-ia)(1-ib)}, \tag{1.45}$$

όπου $a = a(x,y)$ και $b = b(x,y)$. Τότε οι εξισώσεις κίνησης (6.11) για τη $\mathcal{E}$ ισοδυναμούν με τις ακόλουθες εξισώσεις για $a(x,y)$, $b(x,y)$ και $\psi(x,y)$:

$$\begin{aligned}
\nabla^2\psi &= 0,\\
(x-y)a_{,x} &= 2a\left[(xy-1)\psi_{,x} + (1-y^2)\psi_{,y}\right],\\
(x-y)a_{,y} &= 2a\left[-(x^2-1)\psi_{,x} + (xy-1)\psi_{,y}\right],\\
(x+y)b_{,x} &= -2b\left[(xy+1)\psi_{,x} + (1-y^2)\psi_{,y}\right],\\
(x+y)b_{,y} &= -2b\left[-(x^2-1)\psi_{,x} + (xy+1)\psi_{,y}\right].
\end{aligned} \tag{1.46}$$

Επιπλέον, η (6.4) γράφεται ρητά ως

$$\begin{aligned}
\phi_{,x} &= \rho_0^{-1}(x^2-1)^{-1} f^2 \omega_{,y}\ ,\\
\phi_{,y} &= \rho_0^{-1}(y^2-1)^{-1} f^2 \omega_{,x}.
\end{aligned} \tag{1.47}$$

Τότε, οι συναρτήσεις f, γ και ω στη (6.1) δίνονται από:

$$\begin{aligned} f &= e^{2\psi} A B^{-1}, \\ e^{2\gamma} &= K_1 (x^2-1)^{-1} e^{2\gamma_s} A, \\ \omega &= 2\rho_0 e^{-2\psi} A^{-1} C + K_2, \end{aligned} \tag{1.48}$$

όπου

$$\begin{aligned} A &= (x^2-1)(1+ab)^2 - (1-y^2)(b-a)^2, \\ B &= [x+1+(x-1)ab]^2 + [(1+y)a + (1-y)b]^2, \\ C &= (x^2-1)(1+ab)\left[b-a-y(a+b)\right] + (1-y^2)(b-a)\left[1+ab+x(1-ab)\right]. \end{aligned} \tag{1.49}$$

Από την πρώτη των (6.26) βλέπουμε ότι το συστατικό (tt) της μετρικής δίνεται από $g_{tt} = e^{2\psi} A B^{-1}$, οπότε $g_{tt} \approx e^{2\psi}$ για $x \to \infty$. Στην (6.27), K_1 και K_2 είναι σταθερές, ενώ γ_s είναι το δυναμικό γ της αντίστοιχης στατικής μετρικής (6.22).

1.8.1 Η μετρική *Kerr* σε συντεταγμένες *Weyl*

Για $a = b = 0$, το δυναμικό *Ernst* στην (6.23) ανάγεται στο αντίστοιχο (6.12) για τη *Schwarzschild BH*. Θα δείξουμε τώρα ότι, ανάλογα, ανακτούμε τη μετρική *Kerr* από την (6.23) όταν

$$a = -\alpha, \qquad b = \alpha, \qquad \alpha = \sigma\tau\alpha\vartheta. \tag{1.50}$$

Σε αυτή την περίπτωση βρίσκουμε [8]

$$
\begin{aligned}
\mathrm{Re}\{\mathcal{E}\} \equiv f =& \frac{p^2x^2+q^2y^2-1}{(px+1)^2+q^2y^2}, \\
e^{2\gamma} =& \frac{(px)^2+(qy)^2-1}{p^2(x^2-y^2)}, \\
\omega =& -2\rho_0 \frac{q\,(px+1)\,(1-y^2)}{p(p^2x^2+q^2y^2-1)},
\end{aligned} \tag{1.51}
$$

όπου

$$
p=\frac{1-\alpha^2}{1+\alpha^2}, \quad q=\frac{2\alpha}{1+\alpha^2}, \quad p^2+q^2=1. \tag{1.52}
$$

Αντικαθιστώντας επίσης το φανταστικό μέρος της *Ernst* στην (6.25) για να βρούμε την ω και χρησιμοποιώντας τις (6.27) με

$$
K_1=\frac{1}{(1-\alpha^2)^2}, \qquad K_2=-\frac{4\rho_0\alpha}{1-\alpha^2}, \tag{1.53}
$$

καταλήγουμε στη *Kerr* μετρική σε επιμήκεις συντεταγμένες. Η μετάβαση σε συντεταγμένες *Boyer Lindquist* γίνεται από τα υποκατάστατα

$$
\rho_0 x = r-m, \quad y=\cos\theta, \quad \rho_0 = mp, \quad a_0 = mq, \quad \rho_0^2 = m^2-a_0^2, \tag{1.54}
$$

όπου m είναι η μάζα και a_0 η παράμετρος σπιν της *Kerr BH*. Επειδή το σπιν ικανοποιεί $m^2 \geq a_0^2$, από την (6.32) έπεται το εύρος $|\alpha| \leq 1$. Έτσι, ανακτούμε την γνωστή μορφή της *Kerr*:

$$
\begin{aligned}
\mathrm{d}s^2 = &-\left(1-\frac{2mr}{\Sigma}\right)\mathrm{d}t^2+\frac{\Sigma}{\Delta}\mathrm{d}r^2+\Sigma \mathrm{d}\theta^2-\frac{4ma_0 r\sin^2\theta}{\Sigma}\,\mathrm{d}t\,\mathrm{d}\varphi \\
&+\left(r^2+a_0^2+\frac{2ma_0^2 r}{\Sigma}\sin^2\theta\right)\sin^2\theta \mathrm{d}\varphi^2 \ ,
\end{aligned} \tag{1.55}
$$

με

$$\Delta = r^2 - 2mr + a_0^2, \qquad \Sigma = r^2 + a_0^2 \cos^2\theta. \tag{1.56}$$

Συνεπώς, η $Ernst$ (6.23) με a, b όπως στην (6.28) περιγράφει πράγματι τη $Kerr$ σε επιμήκεις σφαιροειδείς συντεταγμένες.

Αν και οι $Boyer\ Lindquist$ είναι γενικώς προτιμητέες για την $Kerr$, ορισμένες εργασίες που ακολουθούν προτιμούν χειρισμούς σε «σφαιρικές» συντεταγμένες τύπου $Weyl$. Οι σφαιρικές (R, u, φ) εκφράζονται μέσω $Weyl$ κανονικών (ρ, z, φ) και $Boyer\ Lindquist$ (r, θ, φ) ως

$$\begin{aligned} R =& \rho_0\sqrt{x^2+y^2-1} = \sqrt{\rho^2+z^2} = \sqrt{(r-m)^2 - \rho_0^2\sin^2\theta}\ , \\ \cos u =& \frac{xy}{\sqrt{x^2+y^2-1}} = \frac{z}{\sqrt{\rho^2+z^2}} = \frac{(r-m)\cos\theta}{\sqrt{(r-m)^2-\rho_0^2\sin^2\theta}}. \end{aligned} \tag{1.57}$$

Οι μεταβάσεις μεταξύ όλων των παραπάνω συστημάτων απλοποιούνται με τις βοηθητικές $R_\pm$:

$$R_\pm(\rho, z) = \sqrt{\rho^2+(z\pm\rho_0)^2} = (r-m) \pm \rho_0\cos\theta = \sqrt{R^2+\rho_0^2 \pm 2\rho_0 R\cos u}\ . \tag{1.58}$$

Με χρήση της (6.32) λαμβάνουμε εύκολα την αντίστροφη μετατροπή της (6.7)

$$\begin{aligned} \rho_0 x &= \frac{1}{2}(R_+ + R_-) = \frac{1}{2}\left(\sqrt{\rho^2+(z+\rho_0)^2} + \sqrt{\rho^2+(z-\rho_0)^2}\right), \\ \rho_0 y &= \frac{1}{2}(R_+ - R_-) = \frac{1}{2}\left(\sqrt{\rho^2+(z+\rho_0)^2} - \sqrt{\rho^2+(z-\rho_0)^2}\right), \end{aligned} \tag{1.59}$$

Στις νέες (R, u, φ), η $Kerr$ γράφεται

$$\mathrm{d}s^2 = f^{-1}\left[e^{2\gamma}(\mathrm{d}R^2 + R^2\mathrm{d}u^2) + R^2\sin^2 u\ \mathrm{d}\varphi^2\right] - f(\mathrm{d}t - \omega\mathrm{d}\varphi)^2, \tag{1.60}$$

όπου

$$
\begin{aligned}
f &= 1 - \frac{4m(R_+ + R_- + 2m)}{(R_+ + R_- + 2m)^2 + \frac{a_0^2}{m^2-a_0^2}(R_+ - R_-)^2}, \\
e^{2\gamma} &= \frac{(R_+ + R_-)^2 - 4m^2 + \frac{a_0^2}{m^2-a_0^2}(R_+ - R_-)^2}{4R_+R_-}, \\
\omega &= -\frac{a_0 m(R_+ + R_- + 2m)(4 - \frac{(R_+-R_-)^2}{(m^2-a_0^2)})}{(R_+ + R_-)^2 - 4m^2 + a_0^2\frac{(R_+-R_-)^2}{(m^2-a_0^2)}}.
\end{aligned}
\tag{1.61}
$$

Τέλος, η (6.38) ανάγεται στην (6.33) μέσω των μετασχηματισμών (6.35) και (6.36).

1.9 Η *Kerr BH* σε εξωτερικά παλιρροϊκά πεδία

Επιθεώρηση των (6.24) δείχνει ότι τα a και b καθορίζονται μόνο μέχρι έναν πολλαπλασιαστικό παράγοντα. Μπορούμε να εκμεταλλευτούμε αυτή την ελευθερία επιλέγοντας σταθερές τιμές όπως στην (6.28). Ας ξαναγράψουμε τη (6.8) ως

$$
\mathrm{d}s^2 = -f\,(\mathrm{d}t - \omega \mathrm{d}\varphi)^2 + h\left(\frac{\mathrm{d}x^2}{x^2-1} + \frac{\mathrm{d}y^2}{1-y^2}\right) + \rho_0^2 f^{-1}(x^2-1)(1-y^2)\mathrm{d}\varphi^2\ , \tag{1.62}
$$

όπου

$$
h = \frac{\rho_0^2}{(1-\alpha^2)^2} B e^{-2\psi + 2V}, \tag{1.63}
$$

$$
V = \gamma_s - \frac{1}{2}\ln\left(\frac{x^2-1}{x^2-y^2}\right), \tag{1.64}
$$

με τις σταθερές K_1, K_2 επιλεγμένες ώστε η (6.40) να ανάγεται στην *Kerr* για $\psi = 0$. Από τις (6.24) βλέπουμε ότι η ψ ικανοποιεί την *Laplace*, η οποία σε (R, u) είναι

$$
\frac{1}{R^2}\partial_R\Big(R^2\partial_R\psi\Big) + \frac{1}{R^2\sin u}\partial_u\Big(\sin u\partial_u\psi\Big) = 0, \tag{1.65}
$$

άρα η γενική λύση σε σφαιρικές αρμονικές $Weyl$ έχει τη μορφή

$$\psi = \sum_{\ell \geq 1} \left(c_\ell R^\ell + \frac{d_\ell}{R^{\ell+1}} \right) P_\ell(\cos u). \tag{1.66}$$

Η λύση αυτή μπορεί επίσης να εκφραστεί σε (x, y) αντικαθιστώντας $R = R(x, y)$ και $u = u(x, y)$ από την (6.35). Τονίζουμε ότι η (6.44) είναι ακριβής λύση της ακριβούς γραμμικής (6.13) εκφρασμένης σε (R, u)—δεν υπάρχει προσέγγιση εδώ. Η σειρά ξεκινά από $\ell = 1$, διότι ο όρος $\ell = 0$ απορροφάται από την παράμετροποίηση της $Ernst$ (6.23). Είναι επίσης σημαντικό να απαιτήσουμε απουσία κωνικών ανωμαλιών. Όπως στην Ενότητα 2, πρέπει $\lim_{\rho \to 0} \gamma = 0$, δηλαδή στις επιμήκεις συντεταγμένες $\lim_{y \to \pm 1} \gamma = 0$. Τότε, απουσία κωνικών ανωμαλιών στον άξονα επιτυγχάνεται αν ισχύει [51]

$$\sum_{n=0}^{\infty} c_{2n+1} = 0 \ . \tag{1.67}$$

Όπως φαίνεται στο [51], η (6.45) προκύπτει από την κανονικότητα της επαγόμενης μετρικής στον ορίζοντα. Υπολογίζοντας τη χαρακτηριστική (Ευλερ) του διδιάστατου συμπαγούς της διατομής του ορίζοντα στο $x = 1$, η (6.45) πρέπει να ισχύει ώστε $\chi = 2$, δηλ. τοπολογικά S^2. Συνεπώς, δεν επιτρέπεται π.χ. μεμονωμένο διπόλο χωρίς συνοδό οκτάπολο.

1.9.1 Η φθίνουσα τετραπολική ιδιομορφή

Η λύση (6.44) είναι άθροισμα φθινουσών (ανάλογων $R^{-\ell-1}$) και αυξανόμενων (ανάλογων R^ℓ) ιδιομορφών. Εξετάζουμε τις τετραπολικές, οπότε

$$\psi = \left(c_2 R^2 + \frac{d_2}{R^3} \right) P_2(\cos u), \tag{1.68}$$

όπου c_2 και d_2 είναι οι εντάσεις αυξανόμενου και φθίνοντος πεδίου, αντίστοιχα. Μπορούμε να υπολογίσουμε τις γενικές εκφράσεις για a, b και V από τις (6.24) και (6.22) για

γ_s

$$
\begin{aligned}
a(x,y) =& -\alpha \exp\Bigg\{2c_2(xy+1)(x-y) - d_2\Big[\left(x^2+y^2-1\right)^{-5/2}\left(2x^5+5x^3\left(y^2-1\right)\right.\\
&\left.-x^2y\left(5y^2-3\right)-3x\left(y^2-1\right)-y\left(2y^4-5y^2+3\right)\right)-2\Big]\Bigg\}, \qquad (1.69)\\
b(x,y) =& \alpha \exp\Bigg\{2c_2(1-xy)(x+y) - d_2\Big[\left(x^2+y^2-1\right)^{-5/2}\left(2x^5+5x^3\left(y^2-1\right)\right.\\
&\left.+x^2y\left(5y^2-3\right)-3x\left(y^2-1\right)+y\left(2y^4-5y^2+3\right)\right)+2\Big]\Bigg\}, \qquad (1.70)\\
V(x,y) =& -\frac{1}{8}\left(y^2-1\right)\Bigg(-2c_2^2x^4\left(9y^2-1\right)+4c_2^2x^2\left(5y^2-1\right)-\frac{24d_2y^2\left(-c_2y^4+c_2y^2+x\right)}{\left(x^2+y^2-1\right)^{5/2}}+\\
&\frac{8d_2\left(x-6c_2y^4\right)}{\left(x^2+y^2-1\right)^{3/2}}+16c_2x+\frac{75d_2^2\left(y^2-1\right)^2y^6}{\left(x^2+y^2-1\right)^6}-\frac{9d_2^2\left(25y^4-38y^2+13\right)y^4}{\left(x^2+y^2-1\right)^5}\\
&+\frac{9d_2^2\left(25y^4-26y^2+5\right)y^2}{\left(x^2+y^2-1\right)^4}-\frac{3d_2^2\left(25y^4-14y^2+1\right)}{\left(x^2+y^2-1\right)^3}\Bigg)+\frac{c_2^2y^4}{4}-\frac{c_2^2y^2}{2}\\
&+\frac{d_2\left(3c_2\left(y^4-1\right)+2x\right)}{\sqrt{x^2+y^2-1}}+V_0. \qquad (1.71)
\end{aligned}
$$

Οι παραπάνω είναι *ακριβείς* εκφράσεις, αποτέλεσμα ολοκλήρωσης των (6.24) με ψ όπως στην (6.44) και αντικατάσταση $R(x,y), u(x,y)$ από (6.35). Η σταθερά V_0 στην (6.49) καθορίζεται από την κανονικότητα [51] $\lim_{y\to\pm1}\gamma(x,y)=0$. Επειδή το d_2 στην (6.46) είναι ανάλογο με τον στατικό TLN για τετραπολικές παλιρροϊκές παραμορφώσεις, εξετάζουμε μόνο τη φθίνουσα ιδιομορφή: θέτοντας $c_2=0$,

$$
\begin{aligned}
\psi(x,y) =& d_2\frac{1}{R^3}P_2(\cos u), \qquad (1.72)\\
a(x,y) =& -\alpha\exp\Big\{-d_2\Big[w(x,y)-w(y,x)\Big]\Big\}, \qquad (1.73)\\
b(x,y) =& \alpha\exp\Big\{-d_2\Big[w(x,y)+w(y,x)\Big]\Big\}, \qquad (1.74)\\
A(x,y) =& 4\alpha^2\exp\Big\{-2d_2w(x,y)\Big\}\Big\{(x^2-1)\sinh^2\Big(d_2w(x,y)-\ln|\alpha|\Big)\\
&+(y^2-1)\cosh^2\Big(d_2w(y,x)\Big)\Big\}, \qquad (1.75)
\end{aligned}
$$

$$
\begin{aligned}
B(x,y) =& 4\alpha^2 \exp\Bigg\{-2d_2 w(x,y)\Bigg\}\Bigg\{\Bigg(\sinh\Big(d_2 w(x,y) - \ln|\alpha|\Big) \\
&+ \cosh\Big(d_2 w(x,y) - \ln|\alpha|\Big)\Bigg)^2 + \Bigg(y\cosh\Big(d_2 w(y,x)\Big) + \sinh\Big(d_2 w(y,x)\Big)\Bigg)^2\Bigg\},
\end{aligned}
\tag{1.76}
$$

$$
\begin{aligned}
C(x,y) =& 4\,\mathrm{sign}\alpha\,\alpha^2 \exp\Bigg\{-2d_2 w(x,y)\Bigg\}\Bigg\{(x^2-1)\sinh\Big(d_2 w(x,y) - \ln|\alpha|\Big)\Bigg(\cosh\Big(d_2 w(y,x)\Big) \\
&+ y\sinh\Big(d_2 w(y,x)\Big)\Bigg) + (1-y^2)\cosh\Big(d_2 w(y,x)\Big)\Bigg(\sinh\Big(d_2 w(x,y) - \ln|\alpha|\Big) \\
&+ x\cosh\Big(d_2 w(x,y) - \ln|\alpha|\Big)\Bigg)\Bigg\},
\end{aligned}
\tag{1.77}
$$

όπου επαναπροσδιορίσαμε $\alpha \mapsto \alpha e^{2d_2}$ και

$$
\begin{aligned}
w(x,y) &= \frac{l(x,y)}{R^5(x,y)}, \\
l(x,y) &= x\Big(2x^4 + (5x^2-3)(y^2-1)\Big)\,.
\end{aligned}
$$

Ενδιαφερόμαστε για πιθανές ανωμαλίες που μπορεί να εισάγει το παλιρροϊκό πεδίο. Ένας τρόπος είναι να ελεγχθεί αν βαθμωτοί καμπυλότητας, όπως ο *Kretschmann*

$$
\mathcal{K} = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}, \tag{1.78}
$$

γίνονται ιδιάζοντες. Καθώς μελετούμε την απόκριση *Kerr BH* σε εξωτερικό πεδίο, δεν θα έπρεπε να εμφανίζονται νέες ανωμαλίες πέραν των γνωστών της *Kerr*· διαφορετικά, θα πρέπει να καλύπτονται από ορίζοντα (όχι γυμνές). Εάν, λοιπόν, η $\mathcal{K}$ καθίσταται ιδιάζουσα αλλού εκτός των γνωστών ιδιομορφιών της *Kerr*, τότε οι στατικοί *Love number* πρέπει να μηδενίζονται, εφόσον αυτές οι νέες ανωμαλίες είναι γυμνές. Η πλήρης έκφραση της $\mathcal{K}$ είναι πολύ ογκώδης. Όμως, αναπτύσσοντας στον ισημερινό ($y=0$) και κοντά στον εξωτερικό ορίζοντα ($x=1$), βρίσκουμε

$$
\mathcal{K}(s,0) \sim \mathcal{K}_0 + e^{\frac{3d_2^2}{16s^3}}\left(\frac{d_2^6}{s^{12}} + \mathcal{O}(s^{-11})\right), \qquad s = x-1, \tag{1.79}
$$

όπου $\mathcal{K}_0$ ο *Kretschmann* της *Kerr* στο $(x = 1, y = 0)$. Βλέπουμε ότι για $d_2 \neq 0$ ο *Kretschmann* αποκλίνει καθώς $s \to 0$, δείχνοντας γυμνή ανωμαλία. Η ανωμαλία αυτή πρέπει να εξαλειφθεί· αυτό επιτυγχάνεται με $d_2 = 0$, άρα ο *Love number* της *Kerr* μηδενίζεται σε οποιαδήποτε τάξη στο εξωτερικό παλιρροϊκό πεδίο.

Θα μπορούσε κανείς να ρωτήσει τι συμβαίνει αν κρατήσουμε και c_2 και d_2 μη μηδενικά. Τότε η $\mathcal{K}$ γίνεται εξαιρετικά μακρά, δυσχεραίνοντας οριστικά συμπεράσματα. Ωστόσο, η $\mathcal{K}$ είναι αναλυτική ως προς c_2, d_2 και μπορεί να αναπτυχθεί σε δυνάμεις τους. Ο όρος ανεξάρτητος του c_2 κοντά στο $x = 1$ δίνεται πάντοτε από την (6.57). Προστίθενται πεπερασμένοι όροι ανάλογοι με θετικές δυνάμεις του c_2. Συνεπώς, η $\mathcal{K}$ θα αποκλίνει για κάθε $d_2 \neq 0$, ανεξαρτήτως c_2.

1.9.2 Η αυξανόμενη τετραπολική ιδιομορφή

Είδαμε ότι η φθίνουσα ιδιομορφή οδηγεί σε γυμνές καμπυλωτικές ανωμαλίες· άρα οι *TLN* της *Kerr* μηδενίζονται στο πλήρως μη γραμμικό επίπεδο. Τώρα μελετάμε την αυξανόμενη τετραπολική ($\ell = 2$) ιδιομορφή. Έχει δειχθεί [48] ότι, εφόσον κρατηθούν μόνο αυξανόμενοι τρόποι (όπως κάνουμε, αφού οι *TLN* μηδενίζονται), μπορούν να υπολογιστούν αναλυτικά οι $a\big(R(x,y), u(x,y)\big)$ και $b\big(R(x,y), u(x,y)\big)$, οπότε και τα μέρη της μετρικής γράφονται ρητώς μέσω πολυωνύμων *Legendre* για αυθαίρετο ℓ [7]:

$$\psi = \sum_{\ell=1}^{\infty} c_\ell \left(\frac{R}{\rho_0}\right)^{\ell} P_\ell(\cos u), \tag{1.80}$$

$$a = -\,\alpha \exp\left\{2\sum_{n=1}^{\infty} c_n \frac{R_-}{\rho_0} \sum_{\ell=0}^{n-1} \left(\frac{R}{\rho_0}\right)^{\ell} P_\ell(\cos u)\right\}, \tag{1.81}$$

$$b = \alpha \exp\left\{2\sum_{n=1}^{\infty} c_n \frac{R_+}{\rho_0} \sum_{\ell=0}^{n-1} (-1)^{n-\ell} \left(\frac{R}{\rho_0}\right)^{\ell} P_\ell(\cos u)\right\}, \tag{1.82}$$

$$\begin{aligned} V = &\sum_{\ell,\ell'=1}^{\infty} \frac{\ell\ell'}{\ell+\ell'} c_\ell c_{\ell'} \left(\frac{R}{\rho_0}\right)^{\ell+\ell'} \Big[P_\ell P_{\ell'} - P_{\ell-1}P_{\ell'-1}\Big] \\ &+ \sum_{\ell=1}^{\infty} c_\ell \sum_{\ell'=0}^{\ell-1} \left[(-1)^{\ell-\ell'+1}\frac{R_+}{\rho_0} - \frac{R_-}{\rho_0}\right] \left(\frac{R}{\rho_0}\right)^{\ell'} P_{\ell'}, \end{aligned} \tag{1.83}$$

$$
\begin{aligned}
h =&\frac{\rho_0^2}{(1-a^2)^2}Be^{2(V-\psi)}, && (1.84)\\
\omega =&2\rho_0 e^{-2\psi}\frac{C}{A}-\frac{4\rho_0\alpha}{1-\alpha^2}\exp\left(-2\sum_{n=0}^{\infty}c_{2n}\right). && (1.85)
\end{aligned}
$$

Για τις τετραπολικές παραμορφώσεις ($\ell = 2$) που μας ενδιαφέρουν, τα δυναμικά ψ, γ_s και V γράφονται

$$
\begin{aligned}
\psi =&c_2\left(\frac{R}{\rho_0}\right)^2 P_2(\cos u), && (1.86)\\
\gamma_s =&\frac{1}{2}\ln\left(\frac{(R_+ + R_-)^2-4\rho_0^2}{4R_+R_-}\right)+c_2^2\left(\frac{R}{\rho_0}\right)^4\Big(P_2^2(\cos u)-P_1^2(\cos u)\Big)\\
&+c_2\left(\frac{R_+}{\rho_0}\left(\frac{R}{\rho_0}\cos u-1\right)-\frac{R_-}{\rho_0}\left(\frac{R}{\rho_0}\cos u+1\right)\right),\\
V =&c_2^2\left(\frac{R}{\rho_0}\right)^4\Big(P_2^2(\cos u)-P_1^2(\cos u)\Big)+c_2\left(\frac{R_+}{\rho_0}\left(\frac{R}{\rho_0}\cos u-1\right)-\frac{R_-}{\rho_0}\left(\frac{R}{\rho_0}\cos u+1\right)\right), && (1.87)
\end{aligned}
$$

και συνεπώς

$$
\begin{aligned}
f =&e^{2\psi}\frac{((R_+ + R_-)^2-4\rho_0^2)(1+ab)^2-(4\rho_0^2-(R_+ - R_-)^2)(b-a)^2}{\left[(R_+ + R_-)(1+ab)+2\rho_0(1-ab)\right]^2+\left[2\rho_0(a+b)+(R_+ - R_-)(a-b)\right]^2}, && (1.88)\\
f^{-1}e^{2\gamma} =&\frac{e^{2(\gamma_s-\psi)}}{(1-a^2)^2}\frac{\left[(R_+ + R_-)(1+ab)+2\rho_0(1-ab)\right]^2+\left[2\rho_0(a+b)+(R_+ - R_-)(a-b)\right]^2}{(R_+ + R_-)^2-4\rho_0^2}, && (1.89)\\
\omega =&e^{-2\psi}\frac{((R_+ + R_-)^2-4\rho_0^2)(1+ab)(2\rho_0(b-a)-(R_+ - R_-)(a+b))}{((R_+ + R_-)^2-4\rho_0^2)(1+ab)^2-(4\rho_0^2-(R_+ - R_-)^2)(b-a)^2} && (1.90)\\
&+\frac{(4\rho_0^2-(R_+ - R_-)^2)(b-a)(2\rho_0(1+ab)+(R_+ + R_-)(1-ab))}{((R_+ + R_-)^2-4\rho_0^2)(1+ab)^2-(4\rho_0^2-(R_+ - R_-)^2)(b-a)^2}-\frac{4\rho_0\alpha}{1-\alpha^2}e^{-2c_2},
\end{aligned}
$$

όπου

$$
a =-\alpha\exp\left\{2c_2\frac{R_-}{\rho_0}\left[1+\frac{R}{\rho_0}\cos u\right]\right\},
$$

$$b = \alpha \exp\left\{2c_2 \frac{R_+}{\rho_0}\left[1 - \frac{R}{\rho_0}\cos u\right]\right\}. \tag{1.91}$$

Τώρα πρέπει να εξετάσουμε αν υπάρχουν γυμνές ανωμαλίες και για την αυξανόμενη ιδιομορφή. Έχει δειχθεί στο [48] ότι για $x > 1$ ανωμαλίες ανακύπτουν όταν $B = 0$ (όπου B όπως στην (6.27». Υποθέτοντας $c_2 < 0$, βρίσκουμε ότι $B \neq 0$, άρα δεν υπάρχουν ανωμαλίες για $x > 1$. Μένει, λοιπόν, ο ορίζοντας $x = 1$. Όπως και πριν, ο υπολογισμός του $Kretschmann$ γύρω από $x = 1$ δείχνει

$$\mathcal{K}(1, y) = \frac{1}{\left(\alpha^2 y^2 + e^{4c_2(y^2-1)}\right)^6} P(y), \tag{1.92}$$

όπου $P(y)$ πολυώνυμο σε y. Από την (6.70) αναλυτικά προκύπτει ότι για $c_2 < 0$ και $y \in [-1, 1]$ ο παρονομαστής δεν μηδενίζεται· άρα δεν προκύπτουν ανωμαλίες στον εξωτερικό ορίζοντα. Η συμπεριφορά (6.70) συμφωνεί με τα διαγράμματα του $Kretschmann$ στο [2]. Αξίζει επίσης ότι η φυσική συνέπεια $c_2 < 0$ είναι η $\epsilon\lambda\acute{\alpha}\tau\tau\omega\sigma\eta$ της γωνιακής ταχύτητας της $Kerr$ υπό παλιρροϊκό πεδίο ($tidalbraking$). Αυτό φαίνεται από την γωνιακή ταχύτητα ορίζοντα σε $Boyer\ Lindquist$ [49]

$$\Omega_H = -\frac{g_{tt}}{g_{t\phi}}\bigg|_H. \tag{1.93}$$

Για το τετραπολικό μας εύρημα,

$$\Omega_H = \frac{a_0}{a_0^2 + r_+^2} e^{2c_2} = \Omega_H^K e^{2c_2}, \tag{1.94}$$

όπου $\Omega_H^K = a_0/(a_0^2 + r_+^2)$ η $Kerr$ τιμή. Άρα, για $c_2 > 0$ η BH «$spin - up$», κάτι μη ρεαλιστικό φυσικά—η παλιρροϊκή πέδηση οδηγεί σε μείωση του σπιν. Συνεπώς, $c_2 < 0$, σε σύμπνοια και με την απουσία ανωμαλιών.

1.10 Ο ρόλος των συμμετριών

Είδαμε ότι οι στατικοί TLN μηδενίζονται ταυτοτικά στο πλήρως μη γραμμικό επίπεδο όχι μόνο για μη περιστρεφόμενη BH [34, 13], αλλά και για περιστρεφόμενες, υποδεικνύοντας υποκείμενη μη γραμμική συμμετρία που εξηγεί τη συμπεριφορά και στις περιστρεφόμενες χωροχρονικές. Η συμμετρία αυτή ήδη εμφανίζεται στο γραμμικό επίπεδο του παλιρροϊκού πεδίου [25, 10, 9, 26, 27, 11, 28, 30, 6, 32, 3, 4, 17, 45]. Για κάθε λύση τρόπου ℓ υπάρχει διατηρούμενη ποσότητα P_ℓ που συνδέεται με τη συμμετρία αυτή. Τα αντίστοιχα φορτία επιτρέπουν «κατάβαση» στο μονοπολικό ($\ell = 0$) μέσω ανυψωτών σκαλοπατιών ($ladderoperators$). Η διατήρηση του P_0 συνεπάγεται αναλλοίωτο P_ℓ για υψηλότερα ℓ, παρέχοντας σκελετό κατανόησης γιατί η φθίνουσα λύση $\sim 1/r^{\ell+1}$ πρέπει να αποκλείεται: συνδέεται με αποκλίσεις στον ορίζοντα. Η μη γραμμική εκδοχή της συμμετρίας έχει ταυτοποιηθεί στα [34, 13].

Καθοριστική παρατήρηση: η εξίσωση για την ψ που διέπει τη στατική διαμόρφωση ακόμη και στο πλήρως μη γραμμικό καθεστώς παραμένει *γραμμική*—είναι η $Laplace$. Συμπτωματικά, είναι η ίδια με εκείνη για στατικό, άμαζο βαθμωτό σε υπόβαθρο $Schwarzschild$ στο γραμμικό επίπεδο. Οι μη γραμμικότητες κωδικοποιούνται στις $a(x, y)$ και $b(x, y)$ που εισέρχονται στην παράμετρο $Ernst$ (6.23). Αναπτύσσοντας [34]

$$\psi(x, y) = \sum_{\ell=0} U_\ell(x) P_\ell(y), \tag{1.95}$$

ορίζουμε τελεστές σκαλοπατιών

$$\begin{aligned} L_\ell^+ &= -(x^2 - 1)\frac{\mathrm{d}}{\mathrm{dx}} - (\ell + 1)x, \\ L_\ell^- &= (x^2 - 1)\frac{\mathrm{d}}{\mathrm{dx}} - \ell x. \end{aligned} \tag{1.96}$$

Αυτοί δρουν ως ανυψωτές/καταβιβαστές πολυπόλων:

$$L_\ell^+ U_\ell \sim U_{\ell+1}, \quad L_\ell^- U_\ell \sim U_{\ell-1}. \tag{1.97}$$

Σημείωση: η δομή ισχύει για τα U_ℓ, όχι για το πλήρες $\mathcal{E}$, καθώς το τελευταίο δεν ικανοποιεί γραμμική εξίσωση. Κατόπιν, κατά το πρότυπο της γραμμικής θεωρίας [26], ορίζουμε διατηρούμενα μεγέθη

$$Q_\ell = (x^2-1)\frac{\mathrm{d}}{\mathrm{dx}}\left(L_1^- L_2^- \cdots L_\ell^-\right) U_\ell, \tag{1.98}$$

για τα οποία:

$$\frac{\mathrm{dQ}_\ell}{\mathrm{dx}} = 0. \tag{1.99}$$

Για τη φθίνουσα λύση, στο άπειρο

$$U_\ell \sim \frac{\beta_\ell}{x^{\ell+1}}, \tag{1.100}$$

οπότε το Q_ℓ παραμένει πεπερασμένο αλλά μη μηδενικό καθώς $x \to \infty$. Κοντά στον ορίζοντα η φθίνουσα λύση αποκλίνει λογαριθμικά ως $\ln(x-1)$. Εφόσον αυξανόμενη και φθίνουσα μοιράζονται το ίδιο Q_ℓ, και η αυξανόμενη είναι πεπερασμένη στον ορίζοντα (άρα $Q_\ell = 0$), η διατήρηση του Q_ℓ απαιτεί τον αποκλεισμό της φθίνουσας λόγω της απόκλισής της. Ωστόσο, απαιτείται πρόσθετο επιχείρημα: λόγω της απόκλισης, η γραμμική θεωρία καταρρέει και πρέπει να εξετασθεί το πλήρες μη γραμμικό πρόβλημα. Εδώ δείξαμε ότι η απόκλιση της φθίνουσας στον ορίζοντα επιβιώνει και μη γραμμικά—ως γυμνή ανωμαλία στο $Kretschmann$. Άρα η φθίνουσα λύση πρέπει να εξαλειφθεί, οδηγώντας σε μηδενικό στατικό *Love number*. Απορρίπτοντας τις φθίνουσες συνιστώσες στην $\psi(x,y)$, δεν μεταφέρονται νέες αποκλίσεις στο μη γραμμικό $\mathcal{E}$, διατηρώντας τη συνέπεια με την $Kerr$.

Η *Laplace* για την ψ στην (6.24) είναι ισοδύναμη δομικά με εκείνη σε διδιάστατο επίπεδο στο αρχικό σύστημα $Weyl$ (ρ, z). Οι λύσεις της γράφονται με ολόμορφες συναρτήσεις

$$\psi(\zeta,\bar{\zeta}) = \Psi(\zeta) + \bar{\Psi}(\bar{\zeta}), \tag{1.101}$$

όπου $\zeta = \rho + iz$. Κάθε αναλυτικός μετασχηματισμός της ζ παράγει νέα λύση. Οι τελεστές σκαλοπατιών είναι γεννήτορες μιας «συμμετρίας» τύπου μορφοδιατήρησης ($conformal$) που σχετίζεται με αυτές τις ολόμορφες μετασχηματίσεις.

Πρέπει, ωστόσο, να σημειωθεί ότι οι παραπάνω (ολο)μορφικές συμμετρίες συνέχονται με τις συμμετρίες του δυναμικού $Ernst$ $\mathcal{E}$. Η επιθεώρηση της δράσης (6.10) ή της (6.11) δείχνει ότι είναι αμφότερες αναλλοίωτες υπό $SL(2,\mathbb{R})$ που δρα πάνω στην $Ernst$ ως

$$\mathcal{E} \to \mathcal{E}' = -i\frac{ai\mathcal{E}+b}{ci\mathcal{E}+d}, \qquad ad-bc=1, \tag{1.102}$$

ή όρους f, ϕ

$$\begin{aligned} \phi \to \phi' =& -\frac{acf^2+(d-c\phi)(b-a\phi)}{c^2f^2+(d-c\phi)^2} \\ f \to f' =& \frac{f}{c^2f^2+(d-c\phi)^2}. \end{aligned} \tag{1.103}$$

Η δράση (6.10) και οι εξισώσεις (6.11) ταυτίζονται με δράση/εξισώσεις ενός μη γραμμικού σ-μοντέλου $SL(2,\mathbb{R})/U(1)$ σε δύο διαστάσεις. Αυτό φαίνεται από την παραμετροποίηση της $SL(2,\mathbb{R})$ με μήτρες 2×2

$$V = \begin{pmatrix} V_-^1 & V_+^1 \\ V_-^2 & V_+^2 \end{pmatrix} = \frac{i}{\sqrt{-2if}} \begin{pmatrix} -\mathcal{E}^* e^{-i\vartheta} & \mathcal{E} e^{i\vartheta} \\ \mathcal{E}^* e^{-i\vartheta} & \mathcal{E}^* e^{i\vartheta} \end{pmatrix}. \tag{1.104}$$

Υπάρχει τοπική $U(1)$ υλοποιούμενη ως μετατοπίσεις $\vartheta \to \vartheta + \Delta\vartheta$, και παγκόσμια $SL(2,\mathbb{R})$ που δρα αριστερά. Επομένως, η $\mathcal{E}$ παραμετροποιεί τον χώρο συζυγιών $SL(2,\mathbb{R})/U(1)$ μόλις καθηλωθεί η τοπική $U(1)$. Τέτοια μη γραμμικά σ-μοντέλα εμφανίζονται συχνά στη GR και είναι γνωστά ως $Ernst$ μοδελς. Εισήχθησαν αρχικά στο πλαίσιο της μείωσης του $Geroch$ [21] και μελετήθηκαν εκτενώς από τον $Ernst$ [18], παρέχοντας πλαίσιο κατανόησης των συμμετριών στάσιμων λύσεων.

Η $SL(2,\mathbb{R})$ συμμετρία του *Ernst*, λόγω μίξης με ευρύτερες (ολο)μορφικές μετασχηματίσεις, γεννά άπειρη άλγεβρα—την άλγεβρα ρευμάτων $SL(2,\mathbb{R})$ άπειρης διάστασης. Οι τελεστές σκαλοπατιών της *Laplace* ανήκουν στο σύνολο γεννητόρων αυτής της άπειρης άλγεβρας, εξηγώντας έτσι τη μηδενικότητα των στατικών TLN για τετραδιάστατες BHs.

Η συμμετρική δομή αυτή είναι χαρακτηριστική των στάσιμων και αξονοσυμμετρικών χωροχρονικών, που ανάγονται εγγενώς σε διδιάστατη δυναμική. Τα *Ernst* μοντέλα και οι συναφείς συμμετρίες έχουν χρησιμοποιηθεί ευρέως για τη μελέτη λύσεων BH, συμπεριλαμβανομένης της παραγωγής ακριβών λύσεων όπως η *Kerr* ή πολυ-BH διαμορφώσεις. Είναι επίσης κρίσιμα στην εξερεύνηση επεκτάσεων της GR, όπου παρόμοια διδιάστατη δυναμική ανακύπτει.

Όλα τα παραπάνω αναδεικνύουν τον πλούτο της συμμετρικής δομής στην διδιάστατη μείωση της GR. Η δομή αυτή, ενδεικτική της άπειρης $SL(2,\mathbb{R})$ άλγεβρας, αποτελεί ισχυρό εργαλείο κατανόησης στάσιμων, αξονοσυμμετρικών λύσεων και υπογραμμίζει τη μηδενικότητα των στατικών παλιρροϊκών αριθμών *Love* [34, 13].

1.11 Συμπεράσματα

Αναλύσαμε τη μη γραμμική παλιρροϊκή απόκριση των *Kerr* BHs υπό εξωτερικά βαρυτικά πεδία. Χρησιμοποιώντας τον φορμαλισμό *Ernst* και συντεταγμένες *Weyl*, επεκτείναμε συστηματικά προηγούμενα αποτελέσματα για *Schwarzschild* BHs στην περιστρεφόμενη περίπτωση. Το κύριο εύρημα είναι ότι οι στατικοί παλιρροϊκοί αριθμοί *Love* των *Kerr* BHs μηδενίζονται σε όλες τις τάξεις του εξωτερικού παλιρροϊκού πεδίου, σε συμφωνία με τις ιδιαίτερες συμμετρίες και ιδιότητες αυτών των χωροχρονικών.

Η μηδενικότητα των στατικών *Love number* αντανακλά την απουσία εσωτερικής δομής στις BHs και την καταλυτική επίδραση των υποκείμενων συμμετριών τους. Σε αντίθεση με τους αστέρες νετρονίων, που εμφανίζουν μη μηδενικούς *Love number* εξαρτώμενους από τη σύστασή τους, οι *BHs* χαρακτηρίζονται από τους ορίζοντές τους και το θεώρημα «$no - hair$». Το αποτέλεσμα υποδηλώνει ότι οι *Kerr BHs* δεν μπορούν να συντηρήσουν κανενός είδους πολυπολική παραμόρφωση ως απόκριση σε στατικές παλιρροϊκές δυνάμεις, ακόμη και λαμβάνοντας υπόψη υψηλότερες μη γραμμικές διορθώσεις. Τονίζει επίσης την ανθεκτικότητα των χωροχρονικών *BH* έναντι παλιρροϊκών διαταραχών, ιδιότητα που τις διακρίνει από άλλα συμπαγή αντικείμενα.

Η ανάλυσή μας ανέδειξε τη χρησιμότητα του δυναμικού *Ernst* στην περιγραφή της συμπεριφοράς των *BHs* σε παλιρροϊκά περιβάλλοντα. Εκφράζοντας την *Kerr* σε συντεταγμένες *Weyl*, μπορέσαμε να γενικεύσουμε την περίπτωση *Schwarzschild* και να εξετάσουμε τον ρόλο της περιστροφής. Η χρήση επιμήκων σφαιροειδών συντεταγμένων διευκόλυνε βασικά αποτελέσματα, επιτρέποντας αυστηρή εξέταση τόσο των αυξανόμενων όσο και των φθινουσών τετραπολικών ιδιομορφών. Η ταυτοποίηση ιδιαζουσών συμπεριφορών στον *Kretschmann* που συνδέονται με τη φθίνουσα ιδιομορφή καθιστά φυσική τη ρύθμιση των *Love number* στο μηδέν. Η προσέγγιση αυτή επαναβεβαιώνει ότι κάθε παλιρροϊκά επαγόμενη ανωμαλία πρέπει να παραμένει κρυμμένη πίσω από ορίζοντα, διαφυλάσσοντας την ακεραιότητα του χωροχρόνου.

Από αστροφυσική σκοπιά, η μηδενικότητα των *Love number* των *Kerr* έχει σημαντικές επιπτώσεις για την αστρονομία *GWs*. Η παλιρροϊκή παραμορφωσιμότητα των *BHs* είναι κρίσιμη παράμετρος στη μοντελοποίηση κυματομορφών από δυαδικά ινσπιραλς, ιδίως σε *BH NS* ή *BH BH*. Η απουσία παλιρροϊκών υπογραφών από *BHs* απλοποιεί τη μοντελοποίηση και παρέχει αυστηρό έλεγχο της *GR* στο ισχυρό πεδίο. Περαιτέρω, τα αποτελέσματα αυτά βελτιώνουν τις θεωρητικές βάσεις ερμηνείας δεδομένων *GWs*, διασφαλίζοντας ότι αποκλίσεις δεν αποδίδονται εσφαλμένα σε μη μοντελοποιημένες παλιρροϊκές επιδράσεις *BH*.

Μελλοντική έρευνα μπορεί να διερευνήσει δυναμικές παλιρροϊκές επιδράσεις (χρονικά εξαρτώμενες διαταραχές, διασπορές/συντονισμοί), κβαντικές διορθώσεις στους *Love number* (όπου η ημι-κλασική βαρύτητα ή θεωρίες χορδών ίσως εισάγουν πρόσθετη δομή), καθώς και παλιρροϊκά φαινόμενα σε υψηλότερες διαστάσεις ή εναλλακτικές θεωρίες βαρύτητας, εξετάζοντας τη γενικότητα των ευρημάτων μας.

Τέλος, σημειώνουμε ότι η λύση μας περιγράφει *στατικές* παλίρροιες και όχι παλιρροϊκή διάχυση. Η τελευταία δεν συλλαμβάνεται από την ακριβή λύση μας, διότι η διάχυση είναι ανάλογη με $(\omega - m\Omega_H)$, όπου ω η συχνότητα της διαταραχής, m ο «μαγνητικός» κβαντικός αριθμός και Ω_H η γωνιακή ταχύτητα στον ορίζοντα. Εφόσον η λύση μας είναι στατική ($\omega = 0$) και αξονοσυμμετρική (χωρίς φ-εξάρτηση, δηλ. $m = 0$), είναι σαφές ότι δεν περιγράφει παλιρροϊκή διάχυση.

Συμπερασματικά, τα αποτελέσματά μας ενισχύουν τη θεμελιώδη φύση των BHs ως γεωμετρικά απλών αλλά βαθιά αινιγματικών αντικειμένων. Η μηδενικότητα των παλιρροϊκών αριθμών $Love$, ακόμη και στο μη γραμμικό καθεστώς, αποτελεί παράδειγμα της αξιοσημείωτης συμμετρίας και αντοχής τους έναντι εξωτερικών διαταραχών. Τα ευρήματα αυτά συμβάλλουν σε βαθύτερη κατανόηση της φυσικής των BHs και του ρόλου τους στον έλεγχο των ορίων της Γενικής Σχετικότητας.

A.K. αναγνωρίζει την υποστήριξη από το $SwissNationalScienceFoundation$ (αριθμός έργου $IZSEZ0\_229414$). O A.P. αναγνωρίζει υποστήριξη από το $SwissNationalScienceFoundation$ (αριθμός έργου $CRSII5\_213497$) καθώς και από το $BoninchiFoundation$ για το έργο «$PBHsintheEraofGWAstronomy$».

Chapter 2

Introduction

2.1 Historical Introduction

2.1.1 The Galilean Relativity

Long before the advent of Special and General Relativity, Galileo had already identified a profound invariance property of Nature. He invites us to imagine ourselves locked in the cabin of a large ship, well below deck, together with a companion and a small laboratory: insects buzzing around, fish swimming in a bowl of water, and a bottle whose water drips steadily into a container placed beneath it. When the ship is anchored and at rest, the situation is entirely unsurprising. The insects fly with no preferred direction, the fish explore the bowl uniformly, the water drops fall vertically into the vessel, and any object we throw to our friend requires the same effort regardless of the direction of the throw, provided the distances are equal. Our own jumps cover equal distances in every horizontal direction.

Galileo then asks us to repeat the same observations when the ship moves at a constant, non–oscillatory velocity on a calm sea. As long as the motion is perfectly uniform, every experiment inside the closed cabin proceeds in precisely the same way as before. No mechanical observation confined to the cabin reveals whether the ship is at rest or in uniform motion. From this thought experiment he distills what we now call the *Galilean principle of relativity*: the outcomes of all mechanical experiments are identical in any two laboratories moving at constant relative velocity. In other words, the laws of mechanics take the same form in all such frames of reference.

A second cornerstone of classical physics due to Galileo is the law of inertia. In modern language it can be stated as follows: a body free of external forces either remains at rest or moves with constant velocity along a straight line. Newton later adopted this as his first law of motion and extended its scope beyond purely mechanical forces. Since the fundamental interactions decrease at least as fast as the inverse square of the distance, a body that is sufficiently remote from all others can be treated, to an excellent approximation, as isolated, and the law of inertia becomes applicable.

From a modern viewpoint, physics describes *events*—occurrences localized in both space and time. Empirically we inhabit a world with three spatial dimensions and one temporal dimension. To assign unique space–time coordinates to an event we introduce a *frame of reference*: an origin, a set of coordinate axes, and a system of synchronized clocks distributed throughout space. A simple example is a Cartesian frame with three mutually orthogonal axes, labelled x, y and z, together with a time coordinate t. An observer at rest with respect to this construction calls it his or her own frame. A frame in which the law of inertia holds is called an *inertial frame of reference*. In classical (pre-relativistic) physics, space and time are assumed to be absolute; all inertial observers agree on time intervals and spatial separations measured in a given frame. In later chapters we shall see how this notion fails in Special Relativity and how Einstein replaced it with a more subtle operational definition based on light signals.

Let us denote by S one inertial frame of reference. The position of an event in S is specified by its coordinates (x, y, z, t). Consider now a second inertial frame S'

moving with respect to S with a constant velocity $\mathbf{V} = V\,\hat{\mathbf{x}}$ along the common x-axis (see Fig. **??**). By convention, the origins of the two frames coincide at $t = t' = 0$. Within Newtonian theory time is universal, so that all properly synchronized clocks in S and S' measure the same time: $t' = t$. Since the relative motion is purely along the x-direction, at an arbitrary instant t the separation between the two origins is

$$OO' = Vt.$$

Therefore the spatial coordinates of a given event in the two frames are related by

$$x' = x - Vt, \qquad y' = y, \qquad z' = z.$$

Collecting space and time together, we obtain the *Galilean transformation*

$$x' = x - Vt, \qquad y' = y, \qquad z' = z, \qquad t' = t. \tag{2.1}$$

This is a linear transformation between the coordinates used by two inertial observers in classical mechanics. Newton's equations of motion are invariant under (2.1); the set of all such transformations constitutes the Galilei group, which encodes the symmetry structure of Newtonian space–time. Special and General Relativity will later replace this structure by Lorentz and diffeomorphism invariance, respectively, but Galilean relativity remains the appropriate framework whenever velocities are small compared to the speed of light and gravitational fields are weak.

2.1.2 Inertia and Inertial Frames of Reference as seen by Newton

From the Newtonian point of view, the dynamical structure of the world is encoded in three empirical statements, Newton's *laws of motion*. Written in modern notation they read

1. A body on which the total external force vanishes remains at rest or moves with constant velocity along a straight line.

2. The time derivative of the momentum of a body equals the total external force acting on it,
$$\frac{d\mathbf{p}}{dt} = \mathbf{F}, \tag{2.2}$$
where $\mathbf{p} = m\mathbf{v}$ is the momentum.

3. Whenever a body A exerts a force $\mathbf{F}_{AB}$ on a body B, the latter exerts an equal and opposite force on the former, $\mathbf{F}_{BA} = -\mathbf{F}_{AB}$.

The first statement is the *law of inertia*. The second law, Eq. (2.2), is often regarded as a *definition* of force, chosen so that the fundamental interactions of Nature can be written in a simple mathematical form. Formally, one might be tempted to view the first law as a special case of the second law with $\mathbf{F} = \mathbf{0}$ and thus redundant. Conceptually, however, the first law does something much more important: it asserts the existence (at least approximately) of special reference frames in which bodies subject to no external influence move with uniform rectilinear motion. These privileged frames are called *inertial* or *Galilean* frames of reference.

If one such frame exists, Galilean relativity implies the existence of an infinite family of them, all moving with constant relative velocities with respect to one another and related by Galilean transformations such as Eq. (2.1). Within this family, Newton's second law has the same functional form, and the notion of an isolated free particle is meaningful.

In practice, strictly inertial frames do not exist in the real universe: every material reference system is subject to gravitational fields and accelerations. The Earth, for example, both rotates about its axis and revolves around the Sun, so an Earth–bound laboratory is not truly inertial. A description of dynamics from the terrestrial point of view must therefore supplement the *real* forces with so–called *fictitious* (or d'Alembert, or inertial) forces, such as the centrifugal and Coriolis forces, to recover an equation of motion of the form $m\mathbf{a} = \mathbf{F}_{\text{real}} + \mathbf{F}_{\text{inertial}}$. By contrast, a heliocentric frame whose origin is placed at the Sun and whose axes are non–rotating with respect to distant galaxies is, to very high accuracy, inertial: the residual accelerations of the solar system barycenter due to neighbouring stars and galactic motion are exceedingly small and can be neglected in most contexts. For this reason one often speaks of motion *with respect to the fixed stars* as shorthand for motion measured in an approximate inertial frame tied to the large–scale distribution of matter in the observable universe.

Galilean Transformations of Velocity and Acceleration

The Galilean transformation of space and time between two inertial frames S and S' moving with relative velocity $\mathbf{V} = V\,\hat{\mathbf{x}}$ along the common x–axis has already been written as

$$x' = x - Vt, \qquad y' = y, \qquad z' = z, \qquad t' = t. \tag{2.3}$$

To obtain the transformation laws for velocities, we differentiate Eq. (2.3):

$$dx' = dx - V\,dt, \qquad dy' = dy, \qquad dz' = dz, \qquad dt' = dt. \tag{2.4}$$

Dividing by $dt' = dt$ we obtain

$$v'_x = \frac{dx'}{dt'} = \frac{dx}{dt} - V = v_x - V, \tag{2.5}$$
$$v'_y = \frac{dy'}{dt'} = \frac{dy}{dt} = v_y, \tag{2.6}$$
$$v'_z = \frac{dz'}{dt'} = \frac{dz}{dt} = v_z, \tag{2.7}$$

or, in compact vector form,

$$\mathbf{v}' = \mathbf{v} - \mathbf{V}. \tag{2.8}$$

It is convenient to rewrite Eq. (2.3) as

$$\mathbf{r}' = \mathbf{r} - \mathbf{V}t, \qquad t' = t, \tag{2.9}$$

with $\mathbf{r}$ and $\mathbf{r}'$ the position vectors in S and S', respectively. Differentiating once yields the velocity transformation (2.8); differentiating once more with respect to time gives the transformation of accelerations,

$$\mathbf{a}' = \mathbf{a}. \tag{2.10}$$

Thus, uniform relative motion between inertial frames does not alter the acceleration of a particle.

To connect with Newton's second law, write

$$\mathbf{F} = m\mathbf{a}, \tag{2.11}$$

with m a frame–independent mass. Using Eq. (2.10) we have $\mathbf{a}' = \mathbf{a}$, and if we further assume that the physical forces are identical in the two frames, $\mathbf{F}' = \mathbf{F}$, then

$$\mathbf{F}' = m\mathbf{a}'. \tag{2.12}$$

Hence the equation of motion retains exactly the same form in all Galilean inertial frames. In group–theoretic language, Newtonian dynamics is invariant under the action of the Galilei group. In General Relativity this role is taken over by local Lorentz invariance and, at a deeper level, by diffeomorphism invariance of the space–time manifold, but the Newtonian notion of inertial frames remains an important limiting case in the regime of weak fields and low velocities.

2.1.3 Maxwell's Equations and the Wave Equation

Within the Newtonian picture of an absolute three–dimensional space evolving in a universal time, the electromagnetic field is described by the four equations of Maxwell. In vacuum they read

$$\nabla\cdot\mathbf{E} = \frac{\rho}{\varepsilon_0}, \qquad \nabla\cdot\mathbf{B} = 0, \qquad \nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}, \qquad \nabla\times\mathbf{B} = \mu_0\mathbf{J} + \mu_0\varepsilon_0\frac{\partial\mathbf{E}}{\partial t}, \quad (2.13)$$

where $\mathbf{E}$ and $\mathbf{B}$ denote the electric and magnetic fields, ρ is the charge density, $\mathbf{J}$ the electric current density, and ε_0 and μ_0 are the electric permittivity and magnetic permeability of empty space.

Each of these relations has a clear physical interpretation in standard vector analysis. The first is Gauss's law for the electric field: the divergence of $\mathbf{E}$ equals the charge density, so that the outward electric flux through an infinitesimal volume element measures the enclosed charge. The second equation expresses Gauss's law for magnetism and is equivalent to the statement that magnetic monopoles do not exist: the magnetic flux through any closed surface vanishes. The third relation is the differential form of Faraday's law of induction; a time–varying magnetic field generates a circulating electric field. Finally, the fourth equation is Ampère's law augmented by Maxwell's displacement current; magnetic fields are produced both by electric currents and by changing electric fields.

In regions of space where there are no free charges and no currents, $\rho = 0$ and $\mathbf{J} = \mathbf{0}$, Maxwell's equations reduce to

$$\nabla\cdot\mathbf{E} = 0, \qquad \nabla\cdot\mathbf{B} = 0, \qquad \nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}, \qquad \nabla\times\mathbf{B} = \mu_0\varepsilon_0\frac{\partial\mathbf{E}}{\partial t}. \qquad (2.14)$$

Combining these equations in the usual way, we obtain a wave equation for the electric field. Taking the curl of Faraday's law and using the identity $\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E}$ together with $\nabla \cdot \mathbf{E} = 0$, we find

$$-\nabla^2 \mathbf{E} = -\frac{\partial}{\partial t}\big(\nabla \times \mathbf{B}\big) = -\mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}. \tag{2.15}$$

Hence

$$\nabla^2 \mathbf{E} = \mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} = \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2}, \tag{2.16}$$

where we have introduced

$$c \equiv \frac{1}{\sqrt{\varepsilon_0 \mu_0}}. \tag{2.17}$$

In Cartesian components this is

$$\frac{\partial^2 \mathbf{E}}{\partial x^2} + \frac{\partial^2 \mathbf{E}}{\partial y^2} + \frac{\partial^2 \mathbf{E}}{\partial z^2} = \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2}. \tag{2.18}$$

A completely analogous manipulation, now starting from Ampère–Maxwell's law and using $\nabla \cdot \mathbf{B} = 0$, leads to the magnetic wave equation

$$\nabla^2 \mathbf{B} = \frac{1}{c^2} \frac{\partial^2 \mathbf{B}}{\partial t^2}. \tag{2.19}$$

Equations (2.16) and (2.19) describe propagating disturbances of the electromagnetic field in empty space—electromagnetic waves—travelling with a phase velocity

$$c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} \simeq 3 \times 10^8 \text{ m/s}, \tag{2.20}$$

a constant which is independent of the motion of the source or of the observer. In the Newtonian context, this universal speed already hints at a deep tension between Maxwell's theory and Galilean kinematics.

The transport of electromagnetic energy is characterized by the *Poynting vector*,

$$\mathbf{S} = \frac{1}{\mu_0}\,\mathbf{E} \times \mathbf{B}, \tag{2.21}$$

which gives the energy flux (power per unit area) carried by the wave.

Maxwell's Theory and the Breakdown of Galilean Invariance

In the nineteenth century it was natural, within Newton's absolute space and time, to assume that electromagnetic waves required a mechanical medium for their propagation, the "luminiferous ether". Even if one could ignore our ignorance of the microphysics of this ether, a more profound issue arises when Maxwell's equations are confronted with the Galilean principle of relativity, according to which the laws of physics should retain their form in all inertial frames related by Galilean transformations.

We have already seen that Newton's second law is invariant under the transformation

$$x' = x - Vt, \qquad y' = y, \qquad z' = z, \qquad t' = t, \tag{2.22}$$

which connects two inertial frames S and S' in relative motion with constant velocity $\mathbf{V} = V\,\hat{\mathbf{x}}$. Maxwell's equations, and in particular the wave equations (2.16)–(2.19), do not share this invariance.

To see this explicitly, consider the scalar wave equation for a single Cartesian component of the electric field,

$$\nabla^2 E = \frac{1}{c^2}\,\frac{\partial^2 E}{\partial t^2}. \tag{2.23}$$

Using the transformation (2.22), the spatial derivative along x is unchanged, $\partial/\partial x =$

$\partial/\partial x'$, whereas the time derivative transforms as

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t'} - V\,\frac{\partial}{\partial x'}. \tag{2.24}$$

Substituting into Eq. (2.23) and expressing everything in terms of the primed coordinates, we obtain

$$\nabla'^2 E = \frac{1}{c^2}\,\frac{\partial^2 E}{\partial t'^2} + \frac{1}{c^2}\left(V^2\,\frac{\partial^2 E}{\partial x'^2} - 2V\,\frac{\partial^2 E}{\partial x'\,\partial t'}\right), \tag{2.25}$$

where ∇'^2 denotes the Laplacian in the primed frame. The additional terms proportional to V and V^2 show that the wave equation in S' does *not* retain the simple form (2.23) unless $V = 0$ or the field configuration is trivial. Thus the Maxwellian wave equation is not invariant under Galilean transformations.

From the Newtonian viewpoint, one way to rescue the "simple" form of Maxwell's equations is to postulate a preferred inertial frame—a rest frame of the ether—in which Eqs. (2.13) and (2.16)–(2.19) hold exactly, while in all other frames more complicated equations such as (2.25) govern the fields. The experimental failure to detect motion relative to such an ether, together with the theoretical tension just described, ultimately led to the abandonment of Newtonian absolute space in favour of a new kinematics: special relativity and its Lorentz symmetry. In General Relativity this insight is elevated further, with the electromagnetic field living on a curved space–time manifold rather than in a fixed Euclidean background.

2.1.4 The Experiment of Michelson and Morley

The most incisive experimental test of the Newtonian–Maxwellian ether picture was carried out by A. A. Michelson and, later, by Michelson and E. W. Morley in the 1880s. Their aim was conceptually simple: if electromagnetic waves propagate in an underlying medium (the luminiferous ether), then the Earth's orbital motion with respect to this medium should influence the observed speed of light. In particular, an interferometric comparison of light propagation along two perpendicular arms should reveal tiny changes as the apparatus is rotated with respect to the "ether wind". The striking failure to detect any such effect became one of the cornerstones in the eventual abandonment of an absolute frame of reference.

The Michelson Interferometer

The 1887 experiment employed an optical interferometer of the type that now bears Michelson's name. The entire device was mounted on a massive stone slab floating in a pool of mercury, so that it could be rotated smoothly about a vertical axis while minimizing mechanical disturbances.

A monochromatic light beam from a source S passes through a narrow slit and is rendered approximately parallel by a converging lens. This beam strikes a half–silvered mirror at 45°, which we denote by a. The mirror a acts as a beam splitter: one part of the incident light is transmitted and sent along a horizontal arm towards a fully reflecting mirror c; the other part is reflected along a perpendicular arm towards a mirror b. After reflection at b and c, the two beams return to the beam splitter a, where they are recombined and directed towards a telescope whose focal plane serves as the observation screen. In the actual experiment the effective optical distance in each arm was increased by multiple reflections between additional mirrors, so that the total path length was an order of magnitude larger than the geometric arm length D.

If the two beams require exactly the same time to traverse their respective paths, they arrive at the telescope in phase and interfere constructively, producing a bright interference fringe. A difference in travel time leads to a phase shift; depending on whether this shift corresponds to an integer or a half–integer multiple of the light period, one observes a bright or a dark fringe at a given point in the field of view. As the apparatus is slowly rotated, any ether–induced anisotropy in the speed of light would manifest itself as a systematic displacement of the fringe pattern.

Classical Analysis in the Ether Frame

Let us analyse the interferometer from the hypothetical ether frame, in which light propagates isotropically with speed c while the entire apparatus moves with constant velocity V along the direction of one of its arms. Denote by D the distance between the beam splitter a and each end mirror as measured in the rest frame of the apparatus. For definiteness, take the arm ac to be aligned with the direction of motion, and the arm ab to be perpendicular to it.

Arm parallel to the ether wind. Consider first the propagation from a to c and back along the parallel arm. During the forward trip, the light pulse moves with speed c while mirror c recedes with speed V. If T_1 is the forward travel time, then the light covers a distance cT_1 while mirror c has moved a distance VT_1; these must add up to the arm length:

$$D + VT_1 = cT_1 \quad \Rightarrow \quad T_1 = \frac{D}{c - V}. \tag{2.26}$$

On the return trip the light travels towards mirror a, which now moves towards the pulse. If T_1' is the return time, we have

$$D - VT_1' = cT_1' \quad \Rightarrow \quad T_1' = \frac{D}{c + V}. \tag{2.27}$$

The total time for the round trip along the parallel arm is therefore

$$T_{\parallel} = T_1 + T_1' = \frac{D}{c - V} + \frac{D}{c + V} = \frac{2Dc}{c^2 - V^2}, \tag{2.28}$$

and the corresponding optical path length is

$$L_1 = cT_\parallel = \frac{2Dc^2}{c^2 - V^2} = \frac{2D}{1 - \dfrac{V^2}{c^2}}. \tag{2.29}$$

Arm perpendicular to the ether wind. For the transverse arm, the light pulse must chase a moving mirror while maintaining a trajectory that keeps it aligned with the arm. In the ether frame the pulse moves with speed c, but its velocity has both a component along the direction of motion of the apparatus and a component perpendicular to it. Let T_2 denote the time for the light to go from a to b. During this interval mirror b drifts a horizontal distance VT_2, while its separation from a in the instantaneous rest frame of the apparatus remains D. Thus the actual path $a \to b$ forms the hypotenuse of a right triangle with sides D and VT_2, so that

$$(cT_2)^2 = D^2 + (VT_2)^2 \quad \Rightarrow \quad T_2 = \frac{D}{\sqrt{c^2 - V^2}}. \tag{2.30}$$

The return trip requires the same time $T_2' = T_2$, because the geometry is symmetric. Hence the total time for the perpendicular arm is

$$T_\perp = T_2 + T_2' = \frac{2D}{\sqrt{c^2 - V^2}}, \tag{2.31}$$

and the corresponding optical path length is

$$L_2 = cT_\perp = \frac{2Dc}{\sqrt{c^2 - V^2}} = \frac{2D}{\sqrt{1 - \dfrac{V^2}{c^2}}}. \tag{2.32}$$

Predicted path difference and fringe shift. The difference in optical path between the two arms is

$$\Delta L = L_1 - L_2 = 2D\left(\frac{1}{1-\dfrac{V^2}{c^2}} - \frac{1}{\sqrt{1-\dfrac{V^2}{c^2}}}\right). \tag{2.33}$$

For velocities much smaller than the speed of light, $V \ll c$, we may expand the two factors:

$$\frac{1}{1-\dfrac{V^2}{c^2}} \simeq 1 + \frac{V^2}{c^2}, \qquad \frac{1}{\sqrt{1-\dfrac{V^2}{c^2}}} \simeq 1 + \frac{1}{2}\frac{V^2}{c^2}, \tag{2.34}$$

which yields, to leading nontrivial order,

$$\Delta L \simeq D\,\frac{V^2}{c^2}. \tag{2.35}$$

A path difference equal to one wavelength λ corresponds to a shift of one fringe in the interference pattern. If we now rotate the whole interferometer by 90°, the roles of the two arms are interchanged and the sign of the path difference reverses. The change in path difference between the two orientations is therefore $2\Delta L$, and the predicted shift of the interference pattern, in units of fringes, is

$$\Delta n = \frac{2\Delta L}{\lambda} \simeq 2\,\frac{V^2}{c^2}\,\frac{D}{\lambda}. \tag{2.36}$$

Taking the Earth's orbital speed to be $V \sim 10^{-4}c$ and inserting the effective arm lengths used by Michelson and Morley (enhanced by multiple reflections so that D/λ was of order 10^6 in the original experiment and ten times larger in the improved version), one obtains a theoretical fringe shift of order $\Delta n \sim 0.4$ when the interferometer is rotated. The experimental apparatus was sensitive to shifts of order 10^{-2} fringes or better, so the predicted effect was comfortably within the detectable range.

Null Result and Its Implications

The actual measurements revealed no such systematic fringe displacement. As the interferometer was slowly rotated through a full 360°, the observed fringes fluctuated irregularly due to mechanical vibrations and environmental effects, but there was no reproducible pattern correlated with the orientation of the apparatus. Interpreted within the ether framework, the data implied an upper bound on the Earth's velocity relative to the ether of only a few kilometres per second — far below the orbital speed and incompatible with the simple Newtonian–Maxwellian picture.

The experiment was repeated many times with increasing sensitivity and under a variety of conditions (different times of day, seasons, wavelengths, and geographical locations), always with essentially null results. The natural conclusion is that the hypothesized luminiferous ether does not exist; there is no preferred inertial frame singled out by electromagnetic phenomena. In this sense the Michelson–Morley experiment marks the empirical demise of Newtonian absolute space and prepares the ground for the relativistic view in which the laws of physics — and, in particular, the speed of light — are the same in all inertial frames.

2.1.5 The Increase of the Mass of the Electron with Speed

The discovery of radioactivity by Becquerel opened a new kinematic regime for charged particles. Electrons emitted in β–decay acquire energies far beyond those achievable at the time with laboratory accelerators, and hence provide an ideal probe of dynamics at velocities comparable with the speed of light. From the Newtonian point of view the inertial mass of a particle is an intrinsic constant, independent of its state of motion. The experiments of Kaufmann at the beginning of the twentieth century gave one of the first clear indications that this conception breaks down at relativistic speeds.

Kaufmann's Arrangement

Kaufmann's setup is sketched in Fig. **??** (a schematic is sufficient for our purposes). Between the poles N and S of a strong electromagnet a uniform magnetic field $\mathbf{B} = B_y\,\hat{\mathbf{y}}$ was established. In the same region, two parallel plates of a capacitor generated a homogeneous electric field $\mathbf{E} = E_y\,\hat{\mathbf{y}}$, directed vertically between the plates. A small sample of radium, placed at a point P, emitted a narrow beam of electrons initially moving along the z–axis, perpendicular to the plane of the figure. The beam thus entered a region where it experienced simultaneously an electric force $e\mathbf{E}$ and a magnetic force $e\mathbf{v} \times \mathbf{B}$. The electric field deflected the electrons in the $\pm y$ direction (towards the positively charged plate), while the magnetic field bent the beam in the x direction. After traversing a distance z the electrons struck a photographic plate parallel to the (x, y)–plane, so that their final positions (x, y) could be recorded; an example of the resulting pattern is shown schematically in Fig. **??** (b). Because the β spectrum of the radium source is continuous, the beam contained electrons with a broad distribution of velocities.

In the Newtonian limit the dynamics in the two transverse directions can be treated independently. An electron entering the fields with speed v along z spends a time

$$t = \frac{z}{v} \tag{2.37}$$

between source and plate. The electric field produces a constant acceleration $a_y = eE_y/m$; the corresponding vertical deflection is

$$y = \frac{1}{2}a_y t^2 = \frac{eE_y}{2m}\,\frac{z^2}{v^2}, \tag{2.38}$$

up to an overall sign determined by the field orientation. The magnetic field bends the trajectory into an arc of a circle of radius $R = mv/(eB_y)$. For small deviations the horizontal displacement at the screen is approximately

$$x \simeq \frac{z^2}{2R} = \frac{eB_y}{2m}\frac{z^2}{v}. \tag{2.39}$$

Eliminating v between Eqs. (2.38) and (2.39), one obtains the classical relation between the deflections,

$$y = -\left(\frac{2mE_y}{eB_y^2 z^2}\right) x^2, \tag{2.40}$$

which is the equation of a parabola in the (x, y) plane. Hence, for a beam with a continuous spectrum of speeds, Newtonian electrodynamics predicts that the impact points of the electrons should lie on a parabolic curve whose shape is determined by the *constant* mass m of the electron.

Kaufmann's photographs, however, showed a markedly different behaviour. When the polarity of the capacitor was reversed, the two sets of data points did not fall on parabolas tangent to the x–axis and to each other, as predicted by Eq. (2.40). Instead, the measured deflections could only be reconciled with the classical formulas if one allowed the effective mass to increase with the electron's speed. This was the first experimental hint that inertial mass is not invariant under changes of velocity when v approaches c.

Electromagnetic Models of the Electron

The observed deviation from Newtonian behaviour prompted a number of theoretical models in which the electron's mass acquires an electromagnetic origin and becomes velocity dependent.

Abraham proposed to treat the electron as a rigid sphere with its charge uniformly distributed over the surface. The field energy of such a charged sphere contributes to the inertia of the particle, and because the field configuration is distorted by motion through the ether, the effective mass depends on v. For the *transverse* inertial mass—

the one probed by Kaufmann's crossed–field geometry—Abraham derived

$$m_A(v) = \frac{3m_0}{4\beta^2}\left[\frac{1+\beta^2}{2\beta}\ln\left(\frac{1+\beta}{1-\beta}\right) - 1\right], \qquad \beta \equiv \frac{v}{c}, \tag{2.41}$$

where m_0 is the low–velocity mass.

Lorentz, on the other hand, adopted a model in which the electron is flattened along the direction of motion according to what later became known as the Lorentz–FitzGerald contraction hypothesis. Under suitable assumptions about the charge distribution he arrived at the much simpler expression

$$m_L(v) = \frac{m_0}{\sqrt{1 - v^2/c^2}}. \tag{2.42}$$

Already in 1903 Lorentz compared this formula with Kaufmann's data and found a surprisingly good agreement.

A third proposal, due to Bucherer and Langevin, considered an electron whose charge is distributed over a spheroid that changes shape but preserves its volume. This led to yet another velocity dependence,

$$m_B(v) = \frac{m_0}{\left(1 - v^2/c^2\right)^{1/3}}. \tag{2.43}$$

Given the experimental uncertainties of the early measurements, it was not initially possible to decide which of the three relations (2.41)–(2.43) was correct.

Precision Measurements and Confirmation of the Lorentz–Einstein Law

More refined experiments were later performed to resolve this ambiguity. In 1909 Bucherer repeated Kaufmann's crossed–field experiment using a more accurate apparatus and again employing β electrons from radium. The deflections in mutually perpendicular electric and magnetic fields allowed him to deduce the ratio $e/m(v)$

as a function of the speed v. Since the charge e of the electron is known, and in particular is independent of v, any variation in this ratio must be attributed solely to changes in the inertial mass. Bucherer's results favoured the Lorentz relation (2.42) over Abraham's, although the experimental precision was still limited.

The decisive measurements were carried out a few years later by Guye and Lavanchy. They produced a monoenergetic electron beam accelerated through a known potential difference, so that the speed of the electrons could be controlled. As in the earlier experiments, combined electric and magnetic fields were used to determine the ratio $m(v)/m(v_0)$ of the mass at speed v to the mass at a low reference speed v_0. With roughly two thousand individual measurements covering the range $0.26c \lesssim v \lesssim 0.48c$, they verified the Lorentz–Einstein expression (2.42) to within about 5×10^{-4} (0.05%). In modern language, the data confirmed the relativistic law

$$m(v) = \gamma(v)\, m_0, \qquad \gamma(v) = \frac{1}{\sqrt{1 - v^2/c^2}}, \tag{2.44}$$

while ruling out the more complicated alternatives.

From the standpoint of Newtonian mechanics, these experiments are profoundly subversive: they demonstrate that the inertial response of a particle cannot be described by a constant mass independent of velocity. Instead, as later understood within the framework of special relativity, the correct dynamical quantity is the four–momentum, whose spatial components reproduce the velocity–dependent behaviour inferred by Kaufmann, Bucherer, Guye and Lavanchy. In the relativistic limit $v \to c$, the effective mass grows without bound, preventing massive particles from ever reaching the speed of light and thereby reconciling the dynamics of matter with the invariant structure of space–time.

2.2 Special Relativity

2.2.1 The Invariance of Maxwell's Equations and the Lorentz Transformation

Between roughly 1892 and 1904, H. A. Lorentz undertook a systematic search for a change of space–time coordinates between two inertial frames, one regarded as "at rest" and the other moving with constant velocity V along a common axis, such that Maxwell's equations retain their form. His early work treated the transformation perturbatively, expanding in powers of V/c and keeping only the lowest–order terms. Larmor later pushed the approximation to include contributions of order $(V/c)^2$. Ultimately, in 1904, Lorentz identified the exact transformation that leaves Maxwell's equations—and hence the electromagnetic wave equation in vacuum—strictly invariant when $\rho = 0$ and $\mathbf{J} = \mathbf{0}$.

This mapping, which supplants the Galilean transformation, relates the coordinates (x, y, z, t) of an event in an inertial frame S to the coordinates (x', y', z', t') in another inertial frame S' moving with constant velocity $\mathbf{V} = V\,\hat{\mathbf{x}}$ relative to S. Lorentz required that the transformation satisfy a number of natural requirements:

1. Free particles must remain free: the world–lines of bodies subject to no forces, which are straight lines in one inertial frame, must again be straight in any other inertial frame. In other words, the transformation must preserve the law of inertia.

2. The notion of coincidence of events must be invariant: if two events occur at the same point in space–time in one inertial frame, they must do so in every inertial frame.

3. Since the relative motion of S and S' is along a fixed line joining their origins, the transformation should respect cylindrical symmetry around this line; transverse

directions orthogonal to $\mathbf{V}$ are equivalent.

4. Most importantly, Maxwell's equations in vacuum, together with the associated wave equation for electromagnetic radiation, must retain their functional form when expressed in the primed coordinates.

In the low–velocity limit $V/c \ll 1$ the transformation ought to reduce to the Galilean one discussed in Sec. **??**. The first two conditions, combined with this requirement, imply that the mapping between coordinates is linear. For motion along the x–axis, Lorentz found that the only transformation consistent with the above criteria is

$$x' = \gamma\,(x - Vt), \qquad y' = y, \qquad z' = z, \qquad t' = \gamma\left(t - \frac{V}{c^2}\,x\right), \tag{2.45}$$

where

$$\gamma \equiv \frac{1}{\sqrt{1 - V^2/c^2}} \tag{2.46}$$

is the usual Lorentz factor. For $V/c \to 0$, $\gamma \to 1$ and Eq. (2.45) indeed collapses to the Galilean transformation.

One can verify by direct substitution that, provided the electric and magnetic fields and the charge and current densities transform appropriately, the homogeneous Maxwell equations and the source-free wave equations retain their form under (2.45). In modern language, the transformation mixes the electric and magnetic fields as different components of a single antisymmetric field tensor, and combines $(\rho, \mathbf{J})$ into a four-current. Lorentz himself succeeded in showing invariance in the vacuum case, but he did not yet possess the full transformation laws for ρ and $\mathbf{J}$ and therefore could not treat configurations with sources in a completely unified way.

This final step was supplied by Poincaré in 1905, who completed the transformation rules for both the space–time coordinates and the electromagnetic quantities so that the *full* Maxwell system is invariant in any inertial frame. Poincaré also recognized that the set of transformations (2.45) forms a group under composition, and he introduced the now-standard terminology of the *Lorentz transformation group*. From the modern viewpoint of General Relativity, this group is understood as the isometry group of Minkowski space–time, encoding the fundamental local symmetry of all relativistic field theories.

2.2.2 The Formulation of the Special Theory of Relativity

By this stage it should be clear that the route from classical mechanics to the Special Theory of Relativity spans several centuries. The story begins with Galileo's formulation of the law of inertia and his recognition that the laws of mechanics admit a relativity principle in uniformly moving frames. During the nineteenth century this insight was repeatedly refined and tested, from Newton's synthesis to Maxwell's field theory, the Michelson–Morley experiment, and the work of Lorentz on invariant transformations. By the dawn of the twentieth century the conceptual ingredients were largely in place; what was missing was a decisive reformulation of space and time themselves.

A particularly important contribution was made by Poincaré. Commenting in 1899 on the negative outcome of the Michelson–Morley experiment, he suggested that optical phenomena likely depend only on the *relative* motion of material systems, light sources and measuring devices, and not on any motion with respect to an absolute ether. By 1900 he openly questioned the existence of the ether and doubted that any physical experiment, no matter how refined, could reveal an "absolute" state of motion. In 1904 he gave a precise formulation of what he called the *principle of relativity*:

> The laws of physical phenomena must be the same for an observer at rest as for an observer in uniform translational motion.

From this he drew the striking conclusion that no signal velocity could exceed that of light in vacuum. Poincaré thus came remarkably close to the modern formulation of Special Relativity, although he did not take the final conceptual step of elevating this principle to the foundation of a new space–time kinematics.

At the turn of the century, faced with the tension between Maxwell's theory and Galilean invariance, three broad logical options presented themselves:

1. Maxwell's equations might be only an approximation; a more fundamental electromagnetic theory, still compatible with Galilean transformations, could replace them.

2. Galileo's relativity principle might hold only for mechanical systems, while electromagnetic phenomena would single out a preferred inertial frame, that of the ether.

3. There might exist a deeper relativity principle governing *all* physical laws—mechanical and electromagnetic alike—requiring a modification of Newtonian mechanics rather than of Maxwell's theory.

Einstein's 1905 paper famously adopted the third alternative. He discarded the ether as superfluous and postulated a single relativistic framework encompassing both dynamics and electrodynamics. The theory he proposed rests on two axioms, which we may state in modern form as

(1) Relativity principle. The laws of physics are identical in all inertial frames of reference.

(2) Constancy of the speed of light. The speed of light in vacuum has the same value c in every inertial frame, regardless of the motion of the source or the observer.

Because these postulates refer explicitly to inertial frames, the theory is called *special* relativity, in contrast with the later general theory in which the relativity principle is extended to arbitrary (accelerated) frames and gravitational fields.

Historical evidence suggests that Einstein did not base his 1905 reasoning directly on the detailed literature of Lorentz and Poincaré, nor was he strongly influenced by the Michelson–Morley experiment in a quantitative way. Rather, he appears to have been guided by the conviction that there is no distinguished inertial frame of reference and by the empirical fact—already implicit in Maxwell's equations—that the speed of light is universal.

In the chapters that follow we shall analyse in detail the kinematical and dynamical consequences of postulates (1) and (2), confronting them with experiment whenever possible. The further generalization of these ideas to spacetimes with gravity, which leads to the General Theory of Relativity, will be considered at a later stage.

2.2.3 The Calibration of a Frame of Reference and the Synchronization of Its Clocks

In relativistic physics we analyse *events*: physical occurrences confined to a sufficiently small region of space and time. To specify the location of an event we assign four coordinates (x, y, z, t), or, in covariant notation, a space–time point x^μ. Special Relativity is precisely the theory of how such events are described in different inertial frames of reference.

Any operational formulation of the theory must explain how an inertial frame is *calibrated*. First, spatial distances must be measurable within that frame using a common standard of length. Second, unlike in classical (Newtonian) physics, time is not an absolute, universal parameter. Each frame must therefore be equipped with an array of identical clocks, distributed throughout space, that are synchronized according to a definite procedure. Only then can we give meaning to the statement that a given event has coordinates (x, y, z, t) in a particular frame.

The two postulates of Special Relativity provide the guiding principles:

1. The laws of physics have the same form in all inertial frames of reference.

2. The speed of light in vacuum has the same value, c, in all inertial frames and is independent of the motion of source and observer.

These postulates suggest natural standards for length and time. A convenient length unit is defined via the wavelength of light associated with a specified atomic transition. Equivalently, one may define the unit of time as the duration of a fixed number of periods of such a monochromatic signal. In modern metrology, the units are chosen so that the numerical value of the vacuum light speed is fixed, $c = 299\,792\,458$ m/s. Given this convention, the metre is effectively defined in terms of c and the second.

Let us now imagine that every spatial point of a given inertial frame carries an ideal clock. The frame is fully calibrated only after these clocks have been synchronized. Einstein proposed the following operational prescription.

Choose one clock, at the origin of the coordinates, as the *reference* clock and call it A. At some reading t_1 of clock A a light pulse is emitted and propagates through vacuum towards another clock, B, located at a fixed spatial separation from A. When the light reaches B, the reading of clock B is denoted by t_B. The pulse is then reflected immediately back towards A, and when it returns the reference clock reads t_2.

We now invoke postulate (ii): the speed of light from A to B equals that from B to A. Consequently, the travel time from A to B is the same as that from B to A, and the event of arrival at B must occur midway between the emission and reception events as measured by clock A. Thus, the instant at which the pulse strikes B is

$$t_{\text{arrive}} = \frac{t_1 + t_2}{2}.$$

To synchronize the two clocks, we simply adjust B so that its reading at the moment of arrival satisfies $t_B = (t_1 + t_2)/2$. The spatial distance from A to B can simultaneously be inferred as

$$D_{AB} = \frac{c\,(t_2 - t_1)}{2},$$

again using the constancy of c. Repeating this procedure for all clocks in the frame yields a synchronized network of clocks and an internally consistent spatial coordinate system.

The existence of such calibrated frames is essential in Special Relativity. Each inertial observer carries along a personal grid of rulers and synchronized clocks which define that observer's space and time. Different inertial frames will not, in general, agree on simultaneity or on the time intervals between events, even though they assign the same invariant space–time structure to physical laws. In General Relativity this idea is extended further: local inertial frames, each equipped with their own synchronized clocks, are patched together on a curved space–time manifold, and the relativity of time between different observers becomes a manifestation of gravitational geometry itself.

2.2.4 The Relativity of Simultaneity

One of the most striking consequences of the invariance of the speed of light is the breakdown of the naive, classical notion of simultaneity. The following thought experiment, formulated in the spirit of Einstein's original argument, makes this completely explicit.

Consider a long train moving with constant velocity V (with $V < c$) relative to an inertial observer standing on a platform. We denote the platform observer by O and the observer at rest with respect to the train by O'. The latter is located at the geometric centre of the train. At the two ends of the train we place identical light sources, labeled A (front) and B (rear). Assume that at the instant when O' passes directly in front of O, two light pulses reach both observers simultaneously: one pulse originated from A and the other from B.

Let us first analyse the situation from the point of view of O', the observer on the train. In his rest frame the train has length L_0, so that A and B are located at equal distances $L_0/2$ from his position at the centre. Because the speed of light is the same in both directions along the train, the pulses travelling from A and from B must have taken equal times to reach O'. Since they *arrive* simultaneously and propagate over equal distances at the same speed c, O' is compelled to conclude that the two pulses were also *emitted* simultaneously in his frame.

Now examine the same events as described by the platform observer O. In his frame the train is moving to the right with speed V, and its length is some value L (not assumed at this stage to be equal to L_0). The events of emission at A and B and of reception at O are all space–time events in this frame. Because light in vacuum travels at speed c relative to O, the arrival time of a pulse at his position is determined solely by the distance between O and the source at the moment of emission.

Suppose first that the rear source B emits its pulse. At that instant the middle of the train (and thus O') has not yet reached O; source B is therefore at a distance $|d_B| > L/2$ from O. Later, the front source A emits its pulse. When this happens O' is still to the left of O, so that A must already be closer to O than the mid–point of the train; hence $|d_A| < L/2$ and consequently $|d_B| > |d_A|$. Since both light pulses propagate with the same speed c towards O, the one that had the larger distance to cover (the pulse from B) must have been emitted *earlier* in O's frame in order for the two pulses to reach him at the same time. In other words, the events "pulse emitted at A" and "pulse emitted at B" are not simultaneous according to O.

We are thus led to a fundamental conclusion. Two events that are simultaneous in one inertial frame (here, the simultaneous arrival of the pulses at O', implying simultaneous emission in the train frame) need not be simultaneous in another inertial frame in relative motion (the platform frame). The postulate that the speed of light has the same value in all inertial frames forces us to abandon the absolute, frame–independent notion of simultaneity that underlies classical mechanics.

It is worth noting that in the figures associated with this thought experiment the length of the train as measured by O' (L_0) and by O (L) are treated as different quantities. This is not an accident: later we shall see that the Lorentz transformation predicts a definite relation between L and L_0 (length contraction). However, the qualitative argument given above does not depend on the explicit form of that relation; it relies only on the constancy of c and on the geometry of the two frames. The relativity of simultaneity is therefore a robust and unavoidable feature of Special Relativity.

2.2.5 Lorentz transformations

We now derive the transformation laws connecting the coordinates of two inertial frames within Special Relativity. Consider two frames, S and S', in standard configuration: their spatial axes are parallel, and S' moves with constant velocity $\mathbf{V} = V\,\hat{\mathbf{x}}$ along the x–axis of S. The origins O and O' coincide at the event $x = x' = y = y' = z = z' = t = t' = 0$.

In Special Relativity physical occurrences are described as *events*, idealized as points in a four–dimensional space–time with coordinates (x, y, z, t) in S and (x', y', z', t') in S'. The worldline of a particle P is then a curve in this four–dimensional manifold. If $(x(t), y(t), z(t))$ denotes the spatial position of P in S, its velocity components are

$$v_x = \frac{dx}{dt}, \qquad v_y = \frac{dy}{dt}, \qquad v_z = \frac{dz}{dt}, \tag{2.47}$$

and similarly, in S' we have

$$v'_x = \frac{dx'}{dt'}, \qquad v'_y = \frac{dy'}{dt'}, \qquad v'_z = \frac{dz'}{dt'}. \tag{2.48}$$

Form of the Transformation

Guided by the relativity principle, we require that uniform rectilinear motion in one inertial frame be described as uniform rectilinear motion in any other. This excludes non–linear coordinate transformations: a non–linear mapping would in general map a straight worldline into a curve, making a freely moving particle appear accelerated in the transformed frame. We therefore assume a linear relation between the coordinates of an event in S and S'. For a boost along the x–axis, spatial isotropy in the directions transverse to $\mathbf{V}$ suggests that y and z should be left unchanged, so we write

$$x' = \alpha x + \varepsilon t, \qquad y' = y, \qquad z' = z, \qquad t' = \delta x + \eta t, \tag{2.49}$$

where α, ε, δ and η are constants depending only on V and c and must be determined from physical requirements.

Taking differentials of (2.49) we find

$$dx' = \alpha\, dx + \varepsilon\, dt, \qquad dy' = dy, \qquad dz' = dz, \qquad dt' = \delta\, dx + \eta\, dt, \tag{2.50}$$

so that the velocity components transform according to

$$v'_x = \frac{dx'}{dt'} = \frac{\alpha v_x + \varepsilon}{\delta v_x + \eta}, \qquad v'_y = \frac{v_y}{\delta v_x + \eta}, \qquad v'_z = \frac{v_z}{\delta v_x + \eta}. \tag{2.51}$$

Kinematical Conditions

First we impose the mutual description of the origins of the two frames. A point at rest in S' has $v'_x = 0$, $v'_y = v'_z = 0$. In S this point is seen to move with velocity V

along x, so we must have $v_x = V$ for that worldline. Substituting in (2.51),

$$0 = v'_x = \frac{\alpha V + \varepsilon}{\delta V + \eta} \quad \Rightarrow \quad \varepsilon = -\alpha V. \tag{2.52}$$

Conversely, a point at rest in S ($v_x = v_y = v_z = 0$) is seen from S' to move with velocity $-V$ along the x'–axis, so $v'_x = -V$. For $v_x = 0$, Eq. (2.51) gives

$$v'_x = \frac{\varepsilon}{\eta} = -V \quad \Rightarrow \quad \frac{\varepsilon}{\eta} = -V. \tag{2.53}$$

Combining (2.52) and (2.53) we conclude

$$\varepsilon = -\alpha V, \qquad \eta = \alpha. \tag{2.54}$$

Thus the unknown coefficients reduce to α and δ.

Invariance of the Speed of Light

The second postulate of Special Relativity asserts that light in vacuum has speed c in every inertial frame. Let us apply this condition to the transformation of velocities. For a light ray in S we have

$$v^2 \equiv v_x^2 + v_y^2 + v_z^2 = c^2. \tag{2.55}$$

In S' the same ray has components given by (2.51), which with (2.54) become

$$v'_x = \frac{\alpha(v_x - V)}{\delta v_x + \alpha}, \qquad v'_y = \frac{v_y}{\delta v_x + \alpha}, \qquad v'_z = \frac{v_z}{\delta v_x + \alpha}. \tag{2.56}$$

The speed in S' must satisfy $v'^2 \equiv v'^2_x + v'^2_y + v'^2_z = c^2$ for *any* light ray. Using (2.56),

$$v'^2 = \frac{\alpha^2(v_x - V)^2 + v_y^2 + v_z^2}{(\delta v_x + \alpha)^2}. \tag{2.57}$$

Equating v'^2 and c^2 and using (2.55) to eliminate $v_y^2 + v_z^2 = c^2 - v_x^2$, we obtain

$$c^2(\delta v_x + \alpha)^2 = \alpha^2(v_x - V)^2 + c^2 - v_x^2, \tag{2.58}$$

which must hold for all possible values of v_x in the range $|v_x| \le c$. Expanding and equating the coefficients of v_x^2, v_x and the constant term yields three algebraic relations:

$$c^2\delta^2 - \alpha^2 + 1 = 0, \tag{2.59}$$

$$2c^2\alpha\delta + 2\alpha^2 V = 0, \tag{2.60}$$

$$c^2\alpha^2 - \alpha^2V^2 - c^2 = 0. \tag{2.61}$$

From (2.61) we find

$$\alpha^2(c^2 - V^2) = c^2 \quad \Rightarrow \quad \alpha = \frac{1}{\sqrt{1 - V^2/c^2}} \equiv \gamma, \tag{2.62}$$

where we have chosen the positive root so that $\alpha \to 1$ as $V \to 0$. Inserting (2.62) into (2.60) then gives

$$\delta = -\gamma\,\frac{V}{c^2}. \tag{2.63}$$

It is straightforward to check that these values also satisfy (2.59). The coefficients of the linear transformation are therefore completely determined:

$$\alpha = \eta = \gamma, \qquad \varepsilon = -\gamma V, \qquad \delta = -\gamma\frac{V}{c^2}. \tag{2.64}$$

Standard Lorentz Transformation

Substituting (2.64) into (2.49) we obtain the *Lorentz transformation* in standard configuration:

$$x' = \gamma(x - Vt), \qquad y' = y, \qquad z' = z, \qquad t' = \gamma\left(t - \frac{V}{c^2}x\right), \tag{2.65}$$

with

$$\gamma \equiv \frac{1}{\sqrt{1 - V^2/c^2}}, \qquad \beta \equiv \frac{V}{c}. \tag{2.66}$$

The transformation is "special" in that it refers to the case where the origins coincide at $t = t' = 0$ and the axes remain parallel.

The inverse transformation is obtained either by solving (2.65) for (x, t), or more simply by exchanging primed and unprimed quantities and replacing V by $-V$:

$$x = \gamma(x' + Vt'), \qquad y = y', \qquad z = z', \qquad t = \gamma\left(t' + \frac{V}{c^2}x'\right). \tag{2.67}$$

Vector Form and Invariance of the Light Cone

For later use it is convenient to write the Lorentz transformation in a coordinate–free form valid for boosts along an arbitrary constant velocity vector $\mathbf{V}$. Let $\mathbf{r} = (x, y, z)$ and $\mathbf{r}' = (x', y', z')$. Decomposing $\mathbf{r}$ into components parallel and orthogonal to $\mathbf{V}$ and using (2.65), one arrives at

$$\mathbf{r}' = \mathbf{r} + (\gamma - 1)\,\frac{\mathbf{V}(\mathbf{V}\cdot\mathbf{r})}{V^2} - \gamma\,\mathbf{V}\,t, \tag{2.68}$$

$$t' = \gamma\left(t - \frac{\mathbf{V}\cdot\mathbf{r}}{c^2}\right). \tag{2.69}$$

The inverse relations are obtained again by replacing $\mathbf{V}$ with $-\mathbf{V}$.

An immediate consequence of (2.68)–(2.69) is the invariance of the light cone.

Suppose that in frame S a spherical wavefront emitted at $t = 0$ satisfies

$$x^2 + y^2 + z^2 - c^2t^2 = 0. \tag{2.70}$$

Inserting the Lorentz transformation into the left–hand side and simplifying one finds

$$x'^2 + y'^2 + z'^2 - c^2t'^2 = 0, \tag{2.71}$$

so that the same wavefront is also spherical in S', expanding with speed c about the origin of that frame.

2.2.6 Measuring Length, Time and adding up velocities

In Special Relativity any statement about "how long" or "how fast" must be anchored to a precise measurement procedure in a specified inertial frame. In what follows we review how spatial lengths, time intervals and relative velocities are operationally defined, and how these notions differ from their Newtonian counterparts.

Length of a Moving Rod and Length Contraction

Consider first a rigid rod aligned with the x–axis. Let the rod be at rest in an inertial frame S, so that its endpoints have fixed spatial coordinates x_A and x_B in this frame. Its *proper length* (or rest length) is then

$$L_0 = x_B - x_A, \tag{2.72}$$

measured by a single observer in S who notes the positions of both ends at the same time t.

Now examine the same rod from another inertial frame S' that moves with constant velocity V in the $+x$ direction relative to S. Observers comoving with the rod in S' wish to determine its length L'; by definition they must record the positions of the two endpoints *simultaneously in their own frame*. Thus, at some fixed time t' in S' the endpoints are at positions x'_A and x'_B, and

$$L' = x'_B - x'_A. \tag{2.73}$$

To relate L' and L_0 we use the inverse Lorentz transformation for a boost along x,

$$x = \gamma(x' + Vt'), \qquad t = \gamma\left(t' + \frac{V}{c^2}x'\right), \tag{2.74}$$

where $\gamma = 1/\sqrt{1 - V^2/c^2}$. Evaluated at the common time t' in S' we have

$$x_A = \gamma(x'_A + Vt'), \qquad x_B = \gamma(x'_B + Vt'). \tag{2.75}$$

Subtracting these expressions gives

$$x_B - x_A = \gamma\big(x'_B - x'_A\big) \quad \Rightarrow \quad L_0 = \gamma L'. \tag{2.76}$$

Hence

$$L' = \frac{L_0}{\gamma} = L_0\sqrt{1 - \frac{V^2}{c^2}}. \tag{2.77}$$

This is the phenomenon of *length contraction*: the length of the rod measured in a frame where it moves with speed V is smaller than its proper length by the factor $1/\gamma$.

No physical compression of the rod is implied; its internal structure need not change. The effect is purely kinematical and arises from the relativity of simultaneity. Indeed, the measurements defining L' are simultaneous in S' ($t'_A = t'_B = t'$), but correspond to different times in S. Using (2.74), the instants at which the endpoints are recorded in S are

$$t_A = \gamma\left(t' + \frac{V}{c^2}x'_A\right), \qquad t_B = \gamma\left(t' + \frac{V}{c^2}x'_B\right), \tag{2.78}$$

so that

$$t_B - t_A = \gamma\frac{V}{c^2}(x'_B - x'_A) = \frac{V}{c^2}L_0. \tag{2.79}$$

From the viewpoint of S, therefore, the front end of the rod is sampled earlier than the rear end while the rod is sliding past, which makes the coordinate distance between the two sampled points smaller than L_0. Length contraction is thus intimately tied to the lack of absolute simultaneity.

Time Intervals and Time Dilation

We now compare time intervals measured by clocks in relative motion. Consider a single clock at rest at position x in frame S. Let two events be the readings t_1 and t_2 indicated by this clock; they occur at the same spatial point in S. The time lapse between them in S is

$$\tau = t_2 - t_1, \tag{2.80}$$

and is called the *proper time* between the two events.

Observers in frame S', moving with velocity V relative to S, assign coordinates (x, t_1) and (x, t_2) to the same events in S, but their own time coordinates are given by the Lorentz transformation

$$t' = \gamma\left(t - \frac{V}{c^2}x\right). \tag{2.81}$$

Thus

$$t'_1 = \gamma\left(t_1 - \frac{V}{c^2}x\right), \qquad t'_2 = \gamma\left(t_2 - \frac{V}{c^2}x\right). \tag{2.82}$$

The interval measured in S' is

$$T' = t'_2 - t'_1 = \gamma(t_2 - t_1) = \gamma\tau. \tag{2.83}$$

Because $\gamma \geq 1$, observers who see the clock in motion register a longer time between its ticks than the clock itself measures. Equivalently, a moving clock appears to "run slow" relative to clocks at rest in the observer's own frame. This is the well-known effect of *time dilation*.

A useful mnemonic is that proper time is always the *shortest* time interval between two fixed events; any other inertial observer, for whom the events occur at different spatial positions, measures a larger time interval.

Adding Velocities Relativistically

Finally, we consider how velocities transform between inertial frames. Let a particle move with velocity components (u'_x, u'_y, u'_z) in frame S', while S' itself moves with speed V along x relative to S. We seek its velocity (u_x, u_y, u_z) in S.

From the Lorentz transformation

$$x' = \gamma(x - Vt), \qquad t' = \gamma\left(t - \frac{V}{c^2}x\right), \tag{2.84}$$

we differentiate with respect to t along the worldline of the particle to obtain

$$u'_x = \frac{dx'}{dt'} = \frac{\gamma(u_x - V)}{\gamma\left(1 - \dfrac{Vu_x}{c^2}\right)} = \frac{u_x - V}{1 - \dfrac{Vu_x}{c^2}}. \tag{2.85}$$

Solving this for u_x yields the *relativistic velocity addition law* in the longitudinal direction:

$$u_x = \frac{u'_x + V}{1 + \dfrac{Vu'_x}{c^2}}. \tag{2.86}$$

Similarly, using the full system of Lorentz transformations and their differentials, the transverse components become

$$u_y = \frac{u'_y}{\gamma\left(1 + \dfrac{V u'_x}{c^2}\right)}, \qquad u_z = \frac{u'_z}{\gamma\left(1 + \dfrac{V u'_x}{c^2}\right)}. \tag{2.87}$$

Equations (2.86) and (2.87) reduce to the familiar Galilean rule $u_x = u'_x + V$ and $u_y = u'_y$, $u_z = u'_z$ in the limit $V \ll c$ and $|u'_x| \ll c$. Crucially, however, if $|u'_x| < c$ and $|V| < c$, then (2.86) guarantees that $|u_x| < c$ as well; the composition of subluminal velocities is always subluminal.

Chapter 3

Differential geometry without a metric

3.1 Some words on Charts, Atlases and differentiable Manifolds

Differentiable manifolds are fundamental constructs in differential geometry. Informally, an n-dimensional manifold $\mathcal{M}$ is a topological space where each point $p \in \mathcal{M}$ possesses a neighborhood $\mathcal{U} \subset \mathcal{M}$ that is homeomorphic to the open unit ball in $\mathbb{R}^n$. For a precise definition, we must first introduce a few essential terms.

A *chart* $(\mathcal{U}, \Phi)$ on $\mathcal{M}$ is defined by an open subset $\mathcal{U} \subset \mathcal{M}$ and a bijective map Φ from $\mathcal{U}$ to an open subset of $\mathbb{R}^n$ (or equivalently, to E^n), such that Φ assigns to each point $p \in \mathcal{U}$ an n-tuple of real numbers called *local coordinates* $(x^1, \ldots, x^n)$. In later computations, we might employ complex conjugate coordinates in place of real ones, although the present discussion will focus solely on real manifolds. (For complex manifolds, refer to works such as Flaherty (1980), and Penrose and Rindler (1984, 1986)).

Two charts $(\mathcal{U}, \Phi)$ and $(\mathcal{U}', \Phi')$ are said to be *compatible* if the transition map

$\Phi' \circ \Phi^{-1}$, defined on the image $\Phi(\mathcal{U} \cup \mathcal{U}')$ of the overlapping region, is a homeomorphism (i.e., a function that is continuous, bijective, and has a continuous inverse). See also Fig. 2.1.

An *atlas* on $\mathcal{M}$ is a collection of mutually compatible charts $\{(\mathcal{U}_\alpha, \Phi_\alpha)\}$ such that every point in $\mathcal{M}$ lies in at least one neighborhood $\mathcal{U}_\alpha$. Often, no single chart suffices to cover the whole manifold; for instance, an n-sphere cannot be covered by a single chart if $n > 0$. An n-dimensional (topological) *manifold* is a topological space $\mathcal{M}$ equipped with an atlas. The manifold $\mathcal{M}$ is called a *differentiable manifold* (of class C^k or analytic) if, for any two overlapping charts, the transition map $\Phi' \circ \Phi^{-1}$ is not only continuous but also differentiable (of class C^k or analytic, respectively).

In this case, the coordinate functions defined on overlapping domains are connected through n differentiable functions of class C^k (or analytic), and the Jacobian matrix of the transformation

$$x^{i'} = x^{i'}(x^j)$$

has non-zero determinant at every point in the intersection:

$$\det\left(\frac{\partial x^{i'}}{\partial x^j}\right) \neq 0. \tag{3.1}$$

A differentiable manifold $\mathcal{M}$ is said to be *orientable* if there exists an atlas for which the Jacobian determinant in Equation (3.1) remains strictly positive on all regions where coordinate charts overlap. This ensures a consistent orientation across the manifold.

Given two manifolds $\mathcal{M}$ and $\mathcal{N}$ of dimensions m and n respectively, one can naturally construct their product manifold $\mathcal{M} \times \mathcal{N}$, which has dimension $m + n$.

Let $\Phi : \mathcal{M} \to \mathcal{N}$ be a map between manifolds. This map is called *differentiable* if, under local charts, the coordinates $(y^1, \dots, y^n)$ in $\mathcal{N}$ are differentiable functions of the coordinates $(x^1, \dots, x^n)$ in $\mathcal{M}$ over open sets $\mathcal{U} \subset \mathcal{M}$ and $\mathcal{V} \subset \mathcal{N}$ where $\Phi(\mathcal{U}) \subset \mathcal{V}$.

If $\Phi(\mathcal{M}) \neq \mathcal{N}$, the image $\Phi(\mathcal{M})$ is termed a *submanifold* of $\mathcal{N}$. More generally, a subset $\mathcal{P} \subset \mathcal{N}$ of dimension $p < n$ is called a submanifold if, near every point of $\mathcal{P}$, there exists a chart $(\mathcal{V}, \Psi)$ of $\mathcal{N}$ such that the local expression of $\mathcal{P}$ within the chart is given by $\mathbb{R}^p \times \{0\}$ inside $\mathbb{R}^n$. In particular, a submanifold of codimension one ($n-1$ dimensional) is referred to as a *hypersurface*.

A *smooth curve* $\gamma(t)$ in $\mathcal{M}$ is defined as a differentiable map from an open interval around the origin in $\mathbb{R}$ into the manifold:

$$\gamma(t) : (-\epsilon, \epsilon) \to \mathcal{M},$$

or occasionally, from a closed interval $[\epsilon, \epsilon]$. A differentiable map $\Phi : \mathcal{M} \to \mathcal{N}$ and its action on such a curve are depicted in Fig. 2.2.

3.2 Vectors, one-forms, Tensors

3.2.1 Vectors

In general, the notion of a vector as an arrow connecting two points loses its validity on a curved manifold. To generalize the concept of vectors from flat Euclidean space to a differentiable manifold $\mathcal{M}$, one defines vectors at a point $p \in \mathcal{M}$ as *tangent vectors*. A tangent vector $\mathbf{v}$ at p is a linear operator acting on the space of smooth functions $f : \mathcal{M} \to \mathbb{R}$, assigning to each function a real number $\mathbf{v}(f)$. This operator satisfies the following axioms:

$$\begin{aligned} &\text{(i)} \quad \mathbf{v}(f+h) = \mathbf{v}(f) + \mathbf{v}(h), \\ &\text{(ii)} \quad \mathbf{v}(fh) = h\mathbf{v}(f) + f\mathbf{v}(h), \\ &\text{(iii)} \quad \mathbf{v}(cf) = c\mathbf{v}(f), \quad c \in \mathbb{R} \text{ constant.} \end{aligned} \tag{3.2}$$

A direct consequence is that $\mathbf{v}(c) = 0$ for any constant function c. Importantly, this definition is coordinate-independent. Geometrically, a tangent vector can be understood as the *directional derivative* of a function f along a smooth curve $\gamma(t)$ passing through p. Expanding f in a Taylor series around p and applying the linearity and Leibniz rule from (3.2), it follows that any tangent vector $\mathbf{v}$ at p can be written in terms of a local coordinate basis as:

$$\mathbf{v} = v^i \frac{\partial}{\partial x^i}. \tag{3.3}$$

The coefficients v^i are called the *components* of $\mathbf{v}$ with respect to the local coordinate system $(x^1, \ldots, x^n)$ around p. According to (3.3), the set of coordinate partial derivatives $\{\partial/\partial x^i\}$ at p spans the tangent space T_p of $\mathcal{M}$, which is a real vector space of dimension n. The set $\{\partial/\partial x^i\}$ is referred to as a *coordinate basis* or *holonomic frame*.

More generally, one can introduce an arbitrary basis $\{e_a\}$ of T_p, where $a = 1, \ldots, n$, not necessarily associated with any coordinate system. Any tangent vector $\mathbf{v} \in T_p$ may then be expressed as a linear combination of these basis vectors:

$$\mathbf{v} = v^a e_a. \tag{3.4}$$

The action of a basis vector e_a on a function f is denoted symbolically by $f_{|a} \equiv e_a(f)$. In coordinate bases, this is often written using a comma notation: $f_{,i} \equiv \partial f/\partial x^i$.

Any nonsingular linear transformation of the basis $\{e_a\}$ leads to a corresponding change in the components v^a of the vector $\mathbf{v}$. Specifically, if

$$\begin{aligned} e_{a'} &= L_{a'}{}^{b} e_b, \\ v^{a'} &= L^{a'}{}_{b} v^b, \\ L^{a'}{}_{b} L_{a'}{}^{c} &= \delta^c_b, \end{aligned} \tag{3.5}$$

then the transformed basis $\{e_{a'}\}$ and the transformed components $v^{a'}$ preserve the vector $\mathbf{v}$.

A coordinate basis $\left\{\frac{\partial}{\partial x^i}\right\}$ is a special case of the general basis $\{e_a\}$. In older literature on general relativity, computations were typically done with respect to coordinate bases. However, in many modern approaches, it is often more convenient to employ a general basis — known as a *frame* or *n-bein* (such as a tetrad or vierbein in four dimensions). When a metric is present, these terms may refer to frames normalized with respect to the metric.

A *vector field* $\mathbf{v}(p)$ on a differentiable manifold $\mathcal{M}$ is a smooth assignment of a tangent vector $\mathbf{v}(p) \in T_p$ to every point $p \in \mathcal{M}$ such that the components v^i in a local coordinate basis are smooth (i.e., differentiable) functions of the coordinates. Thus, a vector field is mathematically described as a smooth map

$$\mathcal{M} \longrightarrow T(\mathcal{M}), \quad p \mapsto \mathbf{v}(p),$$

which defines a *section* of the tangent bundle $T(\mathcal{M})$.

Since vectors are identified with directional derivatives, applying two vector fields in succession to a smooth function f need not commute. In general, the order of application matters. This motivates the definition of the *commutator* (or Lie bracket) $[\mathbf{u}, \mathbf{v}]$ of two vector fields $\mathbf{u}$ and $\mathbf{v}$:

$$[\mathbf{u}, \mathbf{v}](f) = \mathbf{u}(\mathbf{v}(f)) - \mathbf{v}(\mathbf{u}(f)).$$

Let $\{e_a\}$ be a chosen basis of the tangent space. Then, the commutator of two basis vectors is expressed as a linear combination of basis elements:

$$[e_a, e_b] = D^c_{ab} e_c, \qquad D^c_{ab} = -D^c_{ba}, \tag{3.6}$$

where D^c_{ab} are the *structure coefficients* (or commutator coefficients) of the basis. In the case of a coordinate basis $\{\partial/\partial x^i\}$, these coefficients vanish:

$$\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right] = 0.$$

The commutator operation satisfies the *Jacobi identity*:

$$[\mathbf{u}, [\mathbf{v}, \mathbf{w}]] + [\mathbf{v}, [\mathbf{w}, \mathbf{u}]] + [\mathbf{w}, [\mathbf{u}, \mathbf{v}]] = 0, \tag{3.7}$$

for any smooth vector fields $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$. Substituting the expression (3.6) into (3.7), and assuming that the structure coefficients D^c_{ab} are constant (i.e., not functions of position), one obtains the algebraic identity

$$D^f_{d[a} D^d_{bc]} = 0, \tag{3.8}$$

where square brackets denote antisymmetrization over the indices a, b, and c.

3.2.2 One-forms

By definition, a *1-form* (also called a Pfaffian form) $\boldsymbol{\sigma}$ is a linear functional that maps a vector $\mathbf{v} \in T_p$ into a real number. This mapping is called the *contraction* and is denoted by either $\langle \boldsymbol{\sigma}, \mathbf{v} \rangle$ or $\mathbf{v} \lrcorner \boldsymbol{\sigma}$. The contraction operation is linear:

$$\langle \boldsymbol{\sigma}, a\mathbf{u} + b\mathbf{v} \rangle = a\langle \boldsymbol{\sigma}, \mathbf{u} \rangle + b\langle \boldsymbol{\sigma}, \mathbf{v} \rangle, \tag{3.9}$$

for all $a, b \in \mathbb{R}$ and $\mathbf{u}, \mathbf{v} \in T_p$. Similarly, linear combinations of 1-forms $\boldsymbol{\sigma}, \boldsymbol{\tau}$ act linearly:

$$\langle a\boldsymbol{\sigma} + b\boldsymbol{\tau}, \mathbf{v}\rangle = a\langle \boldsymbol{\sigma}, \mathbf{v}\rangle + b\langle \boldsymbol{\tau}, \mathbf{v}\rangle. \tag{3.10}$$

There exist n linearly independent 1-forms ω^a defined by their action on the tangent basis $\{e_b\}$:

$$\langle \omega^a, e_b\rangle = \delta^a_b. \tag{3.11}$$

This set $\{\omega^a\}$ constitutes a basis of the dual space T_p^* of the tangent space T_p, and is said to be dual to the tangent basis $\{e_a\}$. Any 1-form $\boldsymbol{\sigma} \in T_p^*$ can be uniquely written as a linear combination of the basis 1-forms:

$$\boldsymbol{\sigma} = \sigma_a \omega^a, \tag{3.12}$$

where σ_a are real-valued components. Given any $\boldsymbol{\sigma} \in T_p^*$ and $\mathbf{v} \in T_p$, the contraction is evaluated as:

$$\langle \boldsymbol{\sigma}, \mathbf{v}\rangle = \sigma_a v^a. \tag{3.13}$$

The *differential* of a smooth function $f : \mathcal{M} \to \mathbb{R}$ defines a 1-form df through the relation:

$$\langle df, \mathbf{v}\rangle = \mathbf{v}(f) \equiv v^a f_{|a}. \tag{3.14}$$

When $f = x^i$ (i.e., the local coordinates), this implies:

$$\langle dx^i, \frac{\partial}{\partial x^j}\rangle = \delta^i_j, \tag{3.15}$$

which shows that the set $\{dx^i\}$ forms the basis of T_p^* dual to the coordinate basis $\{\partial/\partial x^i\}$ of T_p. Hence, any 1-form $\boldsymbol{\sigma} \in T_p^*$ can be expressed as:

$$\boldsymbol{\sigma} = \sigma_i dx^i. \tag{3.16}$$

In local coordinates, the differential of a function takes the familiar form:

$$df = f_{|a}\omega^a = f_{,i}dx^i. \tag{3.17}$$

From the pointwise definition of 1-forms, we can construct the *1-form bundle* $T^*(\mathcal{M})$, also known as the *cotangent bundle*. A *1-form field* on $\mathcal{M}$ is then a smooth section of this bundle, meaning it assigns to each point $p \in \mathcal{M}$ a 1-form $\boldsymbol{\sigma}(p) \in T_p^*$ such that the components σ_i vary smoothly across the manifold. In tensor calculus, these components are often referred to as the *components of a covariant vector*.

3.2.3 Tensors

A tensor T of type (r, s) at a point p on a manifold $\mathcal{M}$ is an element of the product space

$$T_p^{(r,s)} = \underbrace{T_p \otimes \cdots \otimes T_p}_{r \text{ factors}} \otimes \underbrace{T_p^* \otimes \cdots \otimes T_p^*}_{s \text{ factors}},$$

which means that T is a multilinear map that accepts as arguments r covectors and s vectors, i.e.,

$$T(\sigma^1, \ldots, \sigma^r; v_1, \ldots, v_s),$$

and returns a real number. In particular, the tensor

$$u_1 \otimes \cdots \otimes u_r \otimes \tau^1 \otimes \cdots \otimes \tau^s$$

acts on such a tuple by computing

$$\langle \sigma^1, u_1 \rangle \cdots \langle \sigma^r, u_r \rangle \cdot \tau^1(v_1) \cdots \tau^s(v_s).$$

This mapping is multilinear in all entries. Using a basis $\{e_a\}$ for T_p and a dual basis $\{\omega^b\}$ for T_p^*, any tensor of type (r, s) can be written as a linear combination of tensor products:

$$T = T^{a_1 \cdots a_r}{}_{b_1 \cdots b_s}\ e_{a_1} \otimes \cdots \otimes e_{a_r} \otimes \omega^{b_1} \otimes \cdots \otimes \omega^{b_s}.$$

The components $T^{a_1 \cdots a_r}{}_{b_1 \cdots b_s}$ are the coefficients of the tensor with respect to the chosen basis and are indexed by r contravariant indices and s covariant indices. For general tensors, the order of factors in the tensor product is significant.

A change of basis

$$e_{a'} = L_{a'}{}^{a} e_a, \quad \omega^{a'} = L^{a}{}_{a'} \omega^a, \quad L^{a}{}_{a'} L_{a'}{}^{b} = \delta^b_a,$$

transforms the components of the tensor via the rule:

$$T^{a'_1 \cdots a'_r}{}_{b'_1 \cdots b'_s} = L^{a'_1}{}_{a_1} \cdots L^{a'_r}{}_{a_r} L_{b'_1}{}^{b_1} \cdots L_{b'_s}{}^{b_s} T^{a_1 \cdots a_r}{}_{b_1 \cdots b_s}.$$

In coordinate transformations between coordinate bases $\{\partial/\partial x^a\}$ and $\{\partial/\partial x^{a'}\}$, the matrices take the specific form:

$$L^{a}{}_{a'} = \frac{\partial x^a}{\partial x^{a'}}, \quad L_{a'}{}^{a} = \frac{\partial x^{a'}}{\partial x^a}.$$

Basic algebraic operations involving tensors (e.g., addition, scalar multiplication, tensor product, contraction over a pair of indices, symmetrization and antisymmetrization) are independent of the choice of basis.

3.2.4 Maps of Tensors

Let $\Phi : \mathcal{M} \to \mathcal{N}$ be a smooth map between manifolds. Then any real-valued function f on $\mathcal{N}$ can be pulled back to a function on $\mathcal{M}$ via the pullback map Φ^* defined by

$$\Phi^* f(p) = f(\Phi(p)). \tag{3.18}$$

This induces natural transformations on vectors and 1-forms:

$$\Phi_* : v \in T_p \mapsto \Phi_* v \in T_{\Phi(p)}, \qquad \Phi^* : \sigma \in T^*_{\Phi(p)} \mapsto \Phi^* \sigma \in T^*_p. \tag{3.19}$$

These induced maps satisfy:

(i) The pushforward of a vector satisfies

$$(\Phi_* v)(f)|_{\Phi(p)} = v(\Phi^* f)|_p, \tag{3.20}$$

that is, $\Phi_* v$ is the tangent vector to the curve $\Phi(\gamma(t))$ at $\Phi(p)$, where v is tangent to $\gamma(t)$ at p.

(ii) The pullback preserves contractions:

$$\langle \Phi^* \sigma, v \rangle|_p = \langle \sigma, \Phi_* v \rangle|_{\Phi(p)}. \tag{3.21}$$

From (3.20), one immediately deduces that for any vector fields u, v:

$$[\Phi_* u, \Phi_* v] = \Phi_* [u, v]. \tag{3.22}$$

Let $(x^1, \ldots, x^m)$ and $(y^1, \ldots, y^n)$ be local coordinates in neighborhoods of $p \in \mathcal{M}$ and $\Phi(p) \in \mathcal{N}$, respectively. Then the pullback of a 1-form $\sigma = \sigma_i(y)\, dy^i$ is given by:

$$\Phi^* \sigma = \sigma_i(y(x)) \left(\frac{\partial y^i}{\partial x^k} \right) dx^k = \tilde{\sigma}_k(x)\, dx^k. \tag{3.23}$$

These transformations can be extended to arbitrary tensors of type (r, s) provided that Φ is a diffeomorphism (invertible and smooth). In this case, 3.21 becomes:

$$\langle \Phi^* \sigma, \Phi_*^{-1} v \rangle|_p = \langle \sigma, v \rangle|_{\Phi(p)}. \tag{3.24}$$

Note that Φ^* maps tensors on $\mathcal{N}$ to tensors on $\mathcal{M}$. While the transformation law (3.23) resembles a coordinate change, it actually constructs new tensors (e.g., $\Phi^*\sigma$). This contrasts with basis transformations on $\mathcal{M}$, which only change components while leaving the underlying tensor invariant.

3.3 Exterior products and p-forms

Let $\alpha^1, \ldots, \alpha^p$ be 1-forms. We define a new operation called the *exterior product* (also called the *wedge product*) denoted by $\wedge$, with the following properties:

(i) $\alpha^1 \wedge \cdots \wedge \alpha^p$ is linear in each α^i.

(ii) The product vanishes if any two of the forms coincide.

These axioms imply that the exterior product is totally antisymmetric. In particular, interchanging any two forms introduces a minus sign. From a set of n linearly independent 1-forms $\{\omega^a\}$, we obtain $\binom{n}{p}$ linearly independent p-forms:

$$\omega^{a_1} \wedge \omega^{a_2} \wedge \cdots \wedge \omega^{a_p}, \qquad 1 \leq a_1 < a_2 < \cdots < a_p \leq n, \quad p \leq n. \tag{3.25}$$

By axiom (ii), all such wedge products vanish for $p > n$.

A general p-form α is a linear combination of these elementary p-forms:

$$\alpha = \alpha_{a_1 \cdots a_p}\, \omega^{a_1} \wedge \cdots \wedge \omega^{a_p}. \tag{3.26}$$

In a coordinate basis $\{dx^i\}$, we can write:

$$\alpha = \alpha_{i_1 \cdots i_p}\, dx^{i_1} \wedge \cdots \wedge dx^{i_p}. \tag{3.27}$$

Product of Arbitrary Degree Forms The exterior product extends naturally to forms of arbitrary degree:

$$(\alpha^1 \wedge \cdots \wedge \alpha^p) \wedge (\beta^1 \wedge \cdots \wedge \beta^q) = \alpha^1 \wedge \cdots \wedge \alpha^p \wedge \beta^1 \wedge \cdots \wedge \beta^q. \tag{3.28}$$

This operation is associative and distributive but not strictly commutative. The commutation rule becomes:

$$\alpha_{(p)} \wedge \beta_{(q)} = (-1)^{pq} \beta_{(q)} \wedge \alpha_{(p)}. \tag{3.29}$$

Interior Product (Contraction) Given a vector $\mathbf{v}$ and a p-form α, we define the contraction (interior product) $\mathbf{v} \lrcorner \, \alpha$ to be the $(p-1)$-form:

$$(\mathbf{v} \lrcorner \, \alpha)_{a_2 \ldots a_p} = v^b \alpha_{b a_2 \ldots a_p}, \tag{3.30}$$

which is linear in both $\mathbf{v}$ and α. This operation lowers the degree of a form by 1.

Tensor Interpretation The space of p-forms is naturally identified with antisymmetric covariant tensors of type $(0, p)$. If α and β are p- and q-forms, then the components of their wedge product are given by:

$$(\alpha_{(p)} \wedge \beta_{(q)})_{a_1 \ldots a_{p+q}} = \alpha_{[a_1 \ldots a_p} \beta_{a_{p+1} \ldots a_{p+q}]}. \tag{3.31}$$

For example, the wedge square of a 1-form satisfies:

$$\omega^1 \wedge \omega^2 = \frac{1}{2} \left(\omega^1 \otimes \omega^2 - \omega^2 \otimes \omega^1 \right).$$

Simplicity of p-Forms A p-form α is called *simple* if it can be written as an exterior product of p linearly independent 1-forms:

$$\alpha_{(p)} = \alpha^1 \wedge \alpha^2 \wedge \cdots \wedge \alpha^p. \tag{3.32}$$

3.4 Lie derivatives

Let $\mathbf{v}$ be a vector field on a differentiable manifold $\mathcal{M}$. At each point $p \in \mathcal{M}$, $\mathbf{v}$ defines a unique integral curve $\gamma_p(t)$ such that

$$\gamma_p(0) = p, \quad \dot{\gamma}_p(t) = \mathbf{v}(\gamma_p(t)).$$

This defines a congruence of curves associated with the vector field $\mathbf{v}$. Along each curve $\gamma_p(t)$, the local coordinates $y^i(t)$ satisfy the ordinary differential equations:

$$\frac{dy^i}{dt} = v^i(y^1(t), \ldots, y^n(t)), \tag{3.33}$$

with initial condition $y^i(0) = x^i(p)$.

We now define a one-parameter family of maps Φ_t that drag each point p along the curve $\gamma_p(t)$ to the point $q = \Phi_t(p)$ with coordinates $y^i(t)$. For small values of t, Φ_t is a diffeomorphism. This induces a pullback Φ_t^* acting on tensor fields, and the Lie derivative of a tensor T with respect to $\mathbf{v}$ is defined as:

$$\mathcal{L}_{\mathbf{v}} T \equiv \lim_{t \to 0} \frac{1}{t} \left(\Phi_t^* T - T \right). \tag{3.34}$$

The tensors T and $\Phi_t^* T$ are of the same type and are evaluated at the same point p. Thus, $\mathcal{L}_{\mathbf{v}} T$ is a tensor of the same type at p. The Lie derivative measures the change of T along the flow of $\mathbf{v}$. If T is invariant under the flow, then $\mathcal{L}_{\mathbf{v}} T = 0$.

Using coordinate bases $\{\partial/\partial x^i\}$ at p and $\{\partial/\partial y^j\}$ at $\Phi(p)$, the following relations are used:

$$\left.\frac{\partial y^i}{\partial x^k}\right|_{t=0} = \delta^i_k, \quad \left.\frac{dy^i}{dt}\right|_{t=0} = v^i, \quad \left.\frac{dx^i}{dt}\right|_{t=0} = -v^i. \tag{3.35}$$

We now compute the Lie derivative for various objects:

Function f:

$$\mathcal{L}_{\mathbf{v}} f = v^i f_{,i} = \mathbf{v}(f). \tag{3.36}$$

Proof:

$$\Phi_t^* f|_p = f(y(x,t)), \quad \Rightarrow \quad \mathcal{L}_{\mathbf{v}} f|_p = \left.\frac{\partial f}{\partial y^i}\frac{dy^i}{dt}\right|_p .$$

1-form $\boldsymbol{\sigma} = \sigma_j dx^j$:

$$\mathcal{L}_{\mathbf{v}} \boldsymbol{\sigma} = \left(v^m \sigma_{i,m} + \sigma_m v^m{}_{,i}\right) dx^i. \tag{3.37}$$

Proof:

$$\Phi_t^* \sigma|_p = \sigma_j(y(x,t))\frac{\partial y^j}{\partial x^i} dx^i,$$

$$\mathcal{L}_{\mathbf{v}} \sigma|_p = \left[\frac{\partial \sigma_j}{\partial y^m}\frac{dy^m}{dt}\frac{\partial y^j}{\partial x^i} + \sigma_j \frac{\partial}{\partial x^i}\left(\frac{dy^j}{dt}\right)\right]_{t=0} dx^i.$$

Vector field $\mathbf{u} = u^i \partial_i$:

$$\mathcal{L}_{\mathbf{v}} \mathbf{u} = \left(v^m u^i{}_{,m} - u^m v^i{}_{,m}\right)\frac{\partial}{\partial x^i}. \tag{3.38}$$

Proof:

$$\Phi_t^* u|_p = u^j(y(x,t))\frac{\partial x^i}{\partial y^j}\frac{\partial}{\partial x^i},$$

$$\mathcal{L}_{\mathbf{v}} u|_p = \left[\frac{\partial u^j}{\partial y^m}\frac{dy^m}{dt}\frac{\partial x^i}{\partial y^j} + u^j \frac{\partial}{\partial y^j}\left(\frac{dx^i}{dt}\right)\right]_{t=0} \frac{\partial}{\partial x^i}.$$

Commutator interpretation:

$$\mathcal{L}_{\mathbf{v}}\mathbf{u} = [\mathbf{v}, \mathbf{u}] = v^m \partial_m \left(u^i \partial_i\right) - u^m \partial_m \left(v^i \partial_i\right). \tag{3.39}$$

General Tensor: Using the Leibniz rule and equations (3.37), (3.38), the Lie derivative of a tensor T of type (r, s) is given in components by:

$$(\mathcal{L}_{\mathbf{v}} T)^{i_1 \dots i_r}{}_{j_1 \dots j_s} = v^m \partial_m T^{i_1 \dots i_r}{}_{j_1 \dots j_s} - \sum_{a=1}^{r} T^{i_1 \dots m \dots i_r}{}_{j_1 \dots j_s} \partial_m v^{i_a} + \sum_{b=1}^{s} T^{i_1 \dots i_r}{}_{j_1 \dots m \dots j_s} \partial_{j_b} v^m. \tag{3.40}$$

Lie Bracket Properties: The Jacobi identity and commutator structure imply:

$$\mathcal{L}_{\mathbf{u}} \mathcal{L}_{\mathbf{v}} - \mathcal{L}_{\mathbf{v}} \mathcal{L}_{\mathbf{u}} = \mathcal{L}_{[\mathbf{u}, \mathbf{v}]}. \tag{3.41}$$

Exterior Derivative Commutation: From equations (3.34) and the definition of the exterior derivative, it follows:

$$d(\mathcal{L}_{\mathbf{v}} \alpha) = \mathcal{L}_{\mathbf{v}} (d\alpha), \tag{3.42}$$

for any p-form α. This identity may also be verified by component expressions such as (3.37).

Remarks: The Lie derivative is fundamental in describing geometric symmetries, such as those appearing in general relativity and other physical field theories. It is a natural extension of the directional derivative to tensor fields. The exterior derivative and the Lie derivative are both defined purely from the smooth manifold structure and do not require additional structures like a connection or metric.

However, the Lie derivative depends not just on the value of $\mathbf{v}$ at a point, but on its behavior in a neighborhood. To define derivatives that are local and invariantly defined, such as the covariant derivative, we must introduce additional geometric structure to the manifold, which will be addressed in the next section.

3.5 Covariant derivatives

The covariant derivative $\nabla_{\mathbf{v}}$ in the direction of a vector $\mathbf{v}$ at a point $p \in \mathcal{M}$ maps any tensor to another tensor of the same type. If the direction $\mathbf{v}$ is unspecified, the operator ∇ raises the covariant rank by one; that is, it maps tensors of type (r, s) to tensors of type $(r, s+1)$. In particular, the covariant derivative of a vector $\mathbf{u}$ can be written as:

$$\nabla \mathbf{u} = u^a{}_{;b}\, e_a \otimes \omega^b, \tag{3.43}$$

where the coefficients $u^a{}_{;b}$ are yet to be determined. The directional covariant derivative along $\mathbf{v}$ is then given by:

$$\nabla_{\mathbf{v}} \mathbf{u} = \left(u^a{}_{;b} v^b\right) e_a. \tag{3.44}$$

To define the covariant derivative of the basis vectors e_a in the direction of e_b, we introduce the *connection coefficients* $\Gamma^c{}_{ab}$:

$$\nabla_{e_b} e_a = \Gamma^c{}_{ab}\, e_c. \tag{3.45}$$

To ensure consistency with the Leibniz rule applied to the duality condition

$$\langle \omega^a, e_b \rangle = \delta^a_b, \tag{3.46}$$

the covariant derivative of the dual basis $\{\omega^a\}$ must satisfy:

$$\nabla_{e_b} \omega^a = -\Gamma^a{}_{cb} \omega^c. \tag{3.47}$$

We consider only connections satisfying the symmetry property

$$\nabla_{\mathbf{u}}\mathbf{v} - \nabla_{\mathbf{v}}\mathbf{u} = [\mathbf{u}, \mathbf{v}], \tag{3.48}$$

which is equivalent to the antisymmetry of the connection:

$$2\Gamma^{c}{}_{[ab]} = -D^{c}{}_{ab}, \tag{3.49}$$

with $D^{c}{}_{ab}$ defined as the commutator coefficients from Eq. (3.6). A connection satisfying (3.48) is called torsion-free or symmetric. Using Eq. (3.49), the covariant derivative can be substituted in the expressions for the exterior and Lie derivatives, replacing partial derivatives with covariant derivatives. In particular, the covariant derivative of a tensor

$$T = T^{a_1 \ldots a_r}{}_{b_1 \ldots b_s} e_{a_1} \otimes \cdots \otimes e_{a_r} \otimes \omega^{b_1} \otimes \cdots \otimes \omega^{b_s} \tag{3.50}$$

is given by:

$$\nabla_c u^a = \partial_c u^a + \Gamma^{a}{}_{dc} u^d, \tag{3.51}$$

and the full covariant derivative of T becomes:

$$T^{a_1 \ldots a_r}{}_{b_1 \ldots b_s; c} = \partial_c T^{a_1 \ldots a_r}{}_{b_1 \ldots b_s} + \sum_{i=1}^{r} \Gamma^{a_i}{}_{dc} T^{a_1 \ldots d \ldots a_r}{}_{b_1 \ldots b_s} - \sum_{j=1}^{s} \Gamma^{d}{}_{b_j c} T^{a_1 \ldots a_r}{}_{b_1 \ldots d \ldots b_s}. \tag{3.52}$$

Note that the semicolon notation $u^a{}_{;c} \equiv \nabla_c u^a$ is used.

Coordinate Expression: Let the basis vectors and 1-forms be expressed in terms of the coordinate system:

$$e_a = e^i_a \frac{\partial}{\partial x^i}, \qquad \omega^a = \omega^a{}_i dx^i. \tag{3.53}$$

Then, the connection coefficients become:

$$\Gamma^a{}_{bc} = \omega^a{}_i e^j_b \left(\partial_j e^i_c\right). \tag{3.54}$$

Exterior Derivative of Basis 1-Forms: The exterior derivative of the basis 1-forms ω^a is:

$$d\omega^a = \omega^a{}_{i,j} dx^j \wedge dx^i = -\Gamma^a{}_{bc}\omega^b \wedge \omega^c. \tag{3.55}$$

Introducing the *connection 1-forms*:

$$\Gamma^a{}_b = \Gamma^a{}_{cb}\omega^c, \tag{3.56}$$

Eq. (3.55) becomes the first Cartan structure equation:

$$d\omega^a = -\Gamma^a{}_b \wedge \omega^b. \tag{3.57}$$

Given a connection, the antisymmetric part $\Gamma^a{}_{[bc]}$ can be computed from Eq. (3.57).

Parallel Transport: The covariant derivative defines the notion of parallelism. If $\mathbf{u}(q)$ is the parallel transport of $\mathbf{u}(p)$ along a curve $\gamma(t)$ such that $\gamma(0) = p$, $\gamma(\epsilon) = q$, and $\mathbf{v}(p)$ is tangent to γ at p, then the covariant derivative in direction $\mathbf{v}$ is:

$$\nabla_{\mathbf{v}}\mathbf{u} = \lim_{\epsilon\to 0} \frac{1}{\epsilon}\left(\mathbf{u}(q) - \mathbf{u}_{\|}(q)\right), \tag{3.58}$$

where $\mathbf{u}_{\|}(q)$ is the vector parallel-transported from p to q. A tensor field T is said to be *parallel-transported* along γ if:

$$\nabla_{\mathbf{v}}T = 0. \tag{3.59}$$

In particular, a curve is called an *autoparallel* if its tangent vector is parallel transported along itself:

$$\nabla_{\mathbf{v}}\mathbf{v} = 0. \tag{3.60}$$

3.6 Curvature Tensor

The *curvature tensor* (also called the *Riemann tensor*) is a tensor of type $(1, 3)$ defined by

$$\mathbf{R} = R^a{}_{bcd}\, e_a \otimes \omega^b \otimes \omega^c \otimes \omega^d, \tag{3.61}$$

which assigns a real number to the ordered tuple $(\boldsymbol{\sigma}; \mathbf{w}, \mathbf{u}, \mathbf{v})$, where $\boldsymbol{\sigma}$ is a 1-form and $\mathbf{w}, \mathbf{u}, \mathbf{v}$ are vectors, via

$$\sigma_a w^b u^c v^d R^a{}_{bcd} = \langle \boldsymbol{\sigma}, (\nabla_{\mathbf{u}}\nabla_{\mathbf{v}} - \nabla_{\mathbf{v}}\nabla_{\mathbf{u}} - \nabla_{[\mathbf{u},\mathbf{v}]})\mathbf{w}\rangle. \tag{3.62}$$

By expanding the expression, we find:

$$\sigma_a \left[\left(w^a{}_{;c} v^c\right)_{;d} u^d - \left(w^a{}_{;d} u^d\right)_{;c} v^c \right] = \sigma_a \left(w^a{}_{;cd} - w^a{}_{;dc}\right) v^c u^d. \tag{3.63}$$

Since σ_a, v^c, and u^d are arbitrary, the identity holds:

$$w^a{}_{;cd} - w^a{}_{;dc} = w^b R^a{}_{bdc}, \tag{3.64}$$

which is known as the *Ricci identity*.

From the general formula for covariant derivatives, Eq. (3.52), we obtain the expression for the components of the Riemann tensor:

$$R^a{}_{bcd} = \Gamma^a{}_{bd,c} - \Gamma^a{}_{bc,d} + \Gamma^a{}_{dl}\Gamma^l{}_{bc} - \Gamma^a{}_{cl}\Gamma^l{}_{bd} - D^e{}_{cd}\Gamma^a{}_{be}. \tag{3.65}$$

In a coordinate basis, the last term in Eq. (3.65) vanishes.

The Riemann tensor satisfies the following symmetry relations:

$$R^a{}_{bcd} = -R^a{}_{bdc}, \qquad R^a{}_{[bcd]} = 0. \tag{3.66}$$

The covariant derivatives of the Riemann tensor obey the *Bianchi identities*:

$$R^a{}_{bcd;e} + R^a{}_{bde;c} + R^a{}_{bec;d} = 0. \tag{3.67}$$

By contracting indices, we obtain:

$$R^a{}_{bcd;a} + 2R^a{}_{b[c;d]} = 0, \tag{3.68}$$

and define the *Ricci tensor*:

$$R_{bd} \equiv R^a{}_{bad}. \tag{3.69}$$

Cartan's Structure Equations: An efficient method to compute the curvature components is Cartan's formalism. Define the curvature 2-forms $\Theta^a{}_b$:

$$\Theta^a{}_b = \frac{1}{2} R^a{}_{bcd}\, \omega^c \wedge \omega^d. \tag{3.70}$$

Then, the second Cartan structure equation is:

$$d\Gamma^a{}_b + \Gamma^a{}_c \wedge \Gamma^c{}_b = \Theta^a{}_b, \tag{3.71}$$

where $\Gamma^a{}_b = \Gamma^a{}_{cb}\omega^c$ is the connection 1-form. This formulation allows us to compute curvature directly from the connection.

Finally, the Bianchi identities can be expressed compactly as:

$$d\Theta^a{}_b - \Gamma^a{}_c \wedge \Theta^c{}_b + \Gamma^c{}_b \wedge \Theta^a{}_c = 0. \tag{3.72}$$

Chapter 4

Riemmanian geometry & Einstein field equations

4.1 The metric tensor

A *metric tensor* g on a manifold $\mathcal{M}$ is a smooth field of type $(0,2)$ satisfying:

1. $g : T^0_1 \times T^0_1 \to \mathbb{R}$ is bilinear,
2. $g(X,Y) = g(Y,X)$ for all vectors $X, Y \in T^0_1$ (symmetry),
3. If $g(X,Y) = 0$ for all Y, then $X = 0$ (nondegeneracy).

In a local (noncoordinate) frame $\{e_a\}$ with dual $\{e^a\}$, one writes

$$g = g_{ab}\, e^a \otimes e^b, \quad g_{ab} = g(e_a, e_b) = g_{ba}. \tag{4.1}$$

In a coordinate basis $\{\partial_i\}$ with dual $\{dx^i\}$,

$$g = g_{ij}\, dx^i \otimes dx^j, \quad g_{ij} = g_{ji}. \tag{4.2}$$

Nondegeneracy means $\det[g_{ij}] \neq 0$ everywhere, so the inverse metric g^{ij} exists:

$$g^{ik} g_{kj} = \delta^i{}_j. \tag{4.3}$$

The inverse metric defines a tensor g^{-1} of type $(2, 0)$,

$$g^{-1} = g^{ij}\, \partial_i \otimes \partial_j. \tag{4.4}$$

Using g, the length L of a curve $\lambda\colon [a, b] \to \mathcal{M}$ is

$$L = \int_a^b \sqrt{g_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt}}\; dt, \tag{4.5}$$

so that in differential form $ds^2 = g_{ij}\, dx^i\, dx^j$. Given any (r, s)–tensor T, one may raise and lower indices via contraction with g_{ij} or g^{ij}. For example,

$$T^a{}_{b\cdots} = g^{ai}\, T_{ib\cdots}, \quad T_{iab\cdots} = g_{ij}\, T^j{}_{ab\cdots}. \tag{4.6}$$

The signature of g is the difference between numbers of positive and negative eigenvalues of $[g_{ij}]$ at any point; for a connected manifold, this signature is constant.

Levi–Civita (Christoffel) Connection (from the metric) Requiring a torsion–free connection ∇ compatible with g,

$$\nabla g = 0, \tag{4.7}$$

and

$$\nabla_X Y - \nabla_Y X = [X, Y] \quad \text{(zero torsion)}, \tag{4.8}$$

uniquely determines the Christoffel symbols $\{\Gamma^i_{jk}\}$ by

$$\nabla_{\partial_k}\partial_j = \Gamma^i_{jk}\,\partial_i, \tag{4.9}$$

which in coordinates become the usual formula

$$\Gamma^i_{jk} = \frac{1}{2}\,g^{i\ell}\big(\partial_j g_{k\ell} + \partial_k g_{j\ell} - \partial_\ell g_{jk}\big). \tag{4.10}$$

Properties of the Christoffel Connection .

1. It preserves the inner product under parallel transport: if X, Y are parallel along a curve, then

$$\frac{d}{ds}\big[g(X,Y)\big] = 0. \tag{4.11}$$

2. Geodesics arise as curves whose tangent is parallel–transported along itself:

$$\nabla_{\dot\lambda}\dot\lambda = 0. \tag{4.12}$$

Equivalently, extremizing the length functional (4.5) via the Euler–Lagrange equations yields

$$\frac{d}{ds}\Big(\frac{\partial L}{\partial \dot x^i}\Big) - \frac{\partial L}{\partial x^i} = 0, \tag{4.13}$$

with $L = \sqrt{g_{jk}\,\dot x^j \dot x^k}$. One finds

$$\ddot x^i + \Gamma^i_{jk}\,\dot x^j \dot x^k = 0, \tag{4.14}$$

often written by introducing the “velocity” $u^i = \dot x^i$:

$$u^j \nabla_j u^i = 0. \tag{4.15}$$

Recall that for an affinely parameterized geodesic with tangent $u^i = \dfrac{dx^i}{ds}$ one has

$$u^j \nabla_j u^i = 0. \tag{4.16}$$

Equivalently, in coordinate form,

$$\ddot{x}^i + \Gamma^i_{jk}\, \dot{x}^j \dot{x}^k = 0, \tag{4.17}$$

where dots denote derivatives with respect to the affine parameter s. If one uses a non-affine parameter λ, so that $u^i = dx^i/d\lambda$, the geodesic equation acquires a scale term,

$$u^j \nabla_j u^i = \phi(\lambda)\, u^i, \tag{4.18}$$

for some scalar function $\phi(\lambda)$. Equation (4.18) still implies that the inner product $g_{ij}u^iu^j$ is constant along the curve, consistent with parallel transport.

4.2 Symmetries of the Riemann and Ricci Tensors

For a Levi–Civita connection ($\nabla g = 0$, zero torsion), the Riemann tensor enjoys the additional symmetries

$$g_{im} R^m{}_{jkl} + g_{jm} R^m{}_{ikl} = 0, \tag{4.19}$$

or, with all indices lowered,

$$R_{ijkl} + R_{jikl} = 0. \tag{4.20}$$

Combined with the cyclic (first Bianchi) identity, one deduces

$$R_{ijkl} = R_{klij}. \tag{4.21}$$

Hence the Riemann tensor is antisymmetric in its first and second index pairs and symmetric under interchange of these pairs. The Ricci tensor

$$R_{ij} \equiv R^{k}{}_{ikj} \tag{4.22}$$

is manifestly symmetric:

$$R_{ij} = R_{ji}. \tag{4.23}$$

Using

$$\Gamma^{i}_{ik} = \partial_k \big(\ln \sqrt{|g|}\big), \tag{4.24}$$

one obtains the familiar formula in coordinates:

$$R_{lm} = \partial_i \Gamma^{i}{}_{lm} - \partial_l \Gamma^{i}{}_{im} + \Gamma^{i}{}_{im}\, \Gamma^{k}{}_{kl} - \Gamma^{i}{}_{il}\, \Gamma^{k}{}_{km}, \tag{4.25}$$

where $g = \det[g_{ij}]$.

4.3 Weyl Tensor

The Riemann tensor can be decomposed into its trace (Ricci) part and a trace-free remainder, the *Weyl tensor*:

$$C_{ijkl} = R_{ijkl} - \frac{1}{n-2}\big(g_{ik}R_{jl} + g_{jl}R_{ik} - g_{il}R_{jk} - g_{jk}R_{il}\big) + \frac{1}{(n-1)(n-2)}\, R\,(g_{ik}g_{jl} - g_{il}g_{jk}). \tag{4.26}$$

By construction C_{ijkl} shares all the algebraic symmetries of R_{ijkl} and is fully traceless $g^{ik}C_{ijkl} = 0$. It is conformally invariant under $g \mapsto \Omega^2 g$.

4.4 Four-Dimensional Space–Time

In general relativity, one works on a four-dimensional differentiable manifold with Lorentzian signature $(-,+,+,+)$ (or $(+,-,-,-)$ by convention). Calling t the time coordinate (with c the speed of light), the Minkowski metric reads

$$ds^2 = -c^2\,dt^2 + dx^2 + dy^2 + dz^2. \tag{4.27}$$

All subsequent constructions will assume this four-dimensional signature. On a Lorentzian manifold $(\mathcal{M}, g)$, the metric is no longer positive-definite: for any vector X, the scalar $g(X,X)$ may be positive, zero, or negative. Accordingly, one classifies nonzero vectors into

$$\begin{aligned}&\text{timelike if } g(X,X) < 0,\\ &\text{null (lightlike) if } g(X,X) = 0,\\ &\text{spacelike if } g(X,X) > 0.\end{aligned} \tag{4.28}$$

Material particles of finite rest mass follow timelike geodesics, massless particles follow null geodesics, and spacelike curves cannot be tangent to any physical particle.

Null geodesics remain null under any affine reparametrization (cf. (4.17) and (4.18)), and timelike geodesics satisfy

$$g_{ij}\,u^i u^j = -1, \tag{4.29}$$

where $u^i = dx^i/ds$ is the four-velocity in proper time s.

Lowered Ricci Identity Lowering the Riemann-index in the Ricci identity (3.64) gives

$$R^{\ell}{}_{mji}\,Z_\ell = Z_{i;j;k} - Z_{i;k;j}, \tag{4.30}$$

for any covector Z_i.

Ricci Tensor by Contraction The Ricci tensor arises by contracting the first and third indices of the Riemann tensor:

$$g^{jk} R_{ijkl} = R_{il}. \tag{4.31}$$

Weyl Tensor Revisited The Weyl tensor C_{ijkl} of (4.32) can also be written as

$$C_{ijkl} = R_{ijkl} - \frac{1}{n-2}\big(g_{ik}R_{jl} + g_{jl}R_{ik} - g_{il}R_{jk} - g_{jk}R_{il}\big) + \frac{1}{(n-1)(n-2)}\, R\,(g_{ik}g_{jl} - g_{il}g_{jk}). \tag{4.32}$$

Algebraic and Bianchi Identities In four dimensions ($n = 4$), R_{ijkl} has 20 independent components, while C_{ijkl} and R_{ij} have 10 each. The first Bianchi (cyclic) identity

$$R_{i[jkl]} = 0 \tag{4.33}$$

yields four additional linear relations among components:

$$\begin{aligned} R_{12[34]} - R_{13[24]} + R_{14[23]} &= 0, \\ R_{23[14]} - R_{24[13]} + R_{21[34]} &= 0, \\ R_{34[12]} - R_{31[24]} + R_{32[14]} &= 0, \\ R_{41[23]} - R_{42[31]} + R_{43[12]} &= 0. \end{aligned} \tag{4.34}$$

4.5 The Energy-Momentum tensor

The central idea of general relativity is that gravity is geometry — in particular, curvature of space–time — and that this curvature is sourced by mass–energy (the equivalence of mass and energy). Even in special relativity, valid in a local inertial frame, both the volume and the energy contained within depend on the observer's frame. To encode the energy content of a continuous distribution of matter in a frame-independent way, we require a rank–2 tensor, the *energy–momentum tensor* (or *stress–energy tensor*) $T^{\mu\nu}$, which contains all information about the local energy density, energy flux, momentum density, and stresses.

Invariant volume elements. Let $x^\mu = (x^0, x^i)$ be coordinates on space–time, with x^0 a time coordinate. In a local Lorentz frame the three-volume element is

$$dV = dx^1\, dx^2\, dx^3, \tag{4.35}$$

and the corresponding four-volume element is

$$d\mathcal{V} = dV\, dx^0. \tag{4.36}$$

More invariantly, one writes the proper time element

$$d\tau^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu, \tag{4.37}$$

and the proper four-volume

$$d\Omega = \sqrt{-g}\, d^4x, \tag{4.38}$$

where $g = \det(g_{\mu\nu})$.

Components of $T^{\mu\nu}$. In coordinates x^μ, the tensor $T^{\mu\nu}$ has the following physical interpretations on a hypersurface $x^0 = \text{const}$:

T^{00} : energy density,

T^{0i} : energy flux (or momentum density),

T^{i0} : momentum density (or energy flux in direction i),

T^{ij} : stress tensor (pressures and shear stresses).

Equivalently, one may view $T^{\mu\nu}$ as the flux of the μ–component of four-momentum across a surface of constant x^ν.

Hilbert definition. If the matter action is $S_M = \displaystyle\int \mathcal{L}_M \sqrt{-g}\, d^4x$, then by varying the metric one obtains the stress–energy tensor via

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{\mu\nu}} = -\frac{2}{\sqrt{-g}} \left(\frac{\partial(\mathcal{L}_M \sqrt{-g})}{\partial g^{\mu\nu}} - \partial_\alpha \frac{\partial(\mathcal{L}_M \sqrt{-g})}{\partial(\partial_\alpha g^{\mu\nu})} \right). \tag{4.39}$$

By construction $T_{\mu\nu} = T_{\nu\mu}$ and $\nabla^\mu T_{\mu\nu} = 0$ whenever the matter equations of motion hold.

Ideal fluid. An ideal fluid of rest-frame energy density ρ and isotropic pressure P has

$$T^{\mu\nu} = (\rho + P)\, u^\mu u^\nu + P\, g^{\mu\nu}, \tag{4.40}$$

where u^μ is the fluid's four-velocity ($g_{\mu\nu} u^\mu u^\nu = 1$).

Electromagnetic field. In regions devoid of charges and currents, the Maxwell field strength $F_{\mu\nu}$ contributes

$$T^{\mu\nu} = \frac{1}{4\pi} \Big(F^{\mu\alpha} F^\nu{}_\alpha - \tfrac{1}{4}\, g^{\mu\nu}\, F_{\alpha\beta} F^{\alpha\beta} \Big). \tag{4.41}$$

Klein–Gordon Field Consider a real, spin-0 (scalar) field $\phi(x)$ propagating in a general (curved) space–time and obeying the Klein–Gordon equation. Its stress–energy tensor is given by

$$T_{\mu\nu} = (\nabla_\mu \phi)\,(\nabla_\nu \phi) - \tfrac{1}{2}\, g_{\mu\nu} \big((\nabla\phi)^2 + m^2 \phi^2 \big). \tag{4.42}$$

Local Conservation in Special Relativity In flat space–time (special relativity) where gravity is absent, the energy and momentum of any isolated system satisfy the continuity equations

$$\partial_\mu T^{\mu\nu} = 0, \quad \nu = 0, 1, 2, 3. \tag{4.43}$$

For $\nu = 0$ this states that the rate of change of energy contained in a spatial volume equals the net energy flux through its boundary:

$$\partial_0 T^{00} + \partial_i T^{i0} = \partial_0 \rho + \nabla\!\cdot\!\mathbf{S} = -\oint_{\partial V} T^{i0}\, d\Sigma_i, \tag{4.44}$$

where $\rho = T^{00}$ is the energy density and $S^i = T^{i0}$ the energy flux (or momentum density). Similarly, for $\nu = j$ one obtains the momentum-continuity law,

$$\partial_0 T^{0j} + \partial_i T^{ij} = 0 \quad \Longleftrightarrow \quad \frac{d}{dt}\int_V T^{0j}\, d^3x = -\oint_{\partial V} T^{ij}\, d\Sigma_i. \tag{4.45}$$

If $T^{\mu\nu} = 0$ outside the region V, integrating over V shows that the total energy–momentum

$$P^\nu \equiv \int_V T^{0\nu}\, d^3x \tag{4.46}$$

is constant ($\dot{P}^\nu = 0$) whenever no flux escapes through the boundary.

General Curved Space–Time In a general curved manifold, the local conservation law becomes

$$\nabla_\mu T^{\mu\nu} = 0, \tag{4.47}$$

which in coordinates reads

$$\partial_\mu T^{\mu\nu} + \Gamma^\mu{}_{\mu\lambda}\, T^{\lambda\nu} + \Gamma^\nu{}_{\mu\lambda}\, T^{\mu\lambda} = 0. \tag{4.48}$$

Using $\Gamma^\mu{}_{\mu\lambda} = \partial_\lambda \ln\sqrt{-g}$, one can rewrite this as

$$\frac{1}{\sqrt{-g}}\,\partial_\mu\big(\sqrt{-g}\, T^{\mu\nu}\big) + \Gamma^\nu{}_{\mu\lambda}\, T^{\mu\lambda} = 0. \tag{4.49}$$

4.6 Einstein Equations

The Einstein field equations are the fundamental equations of general relativity, since they relate the geometry (curvature) of a space–time to the distribution of matter and energy within it. They were postulated axiomatically by A. Einstein in 1915 and take the form

$$R_{\mu\nu} - \tfrac{1}{2}\, g_{\mu\nu}\, R \;=\; 8\pi G\, T_{\mu\nu}\,, \tag{4.50}$$

where G is Newton's gravitational constant.

Introducing the Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu} R$, one writes equivalently

$$G_{\mu\nu} \;=\; 8\pi G\, T_{\mu\nu}. \tag{4.51}$$

Thus the matter–energy sources contained in a space–time (represented by the stress–energy tensor $T_{\mu\nu}$) determine its curvature (encoded by $G_{\mu\nu}$). The field equations form a system of ten coupled, nonlinear, second-order partial differential equations for the metric components $g_{\mu\nu}(x)$.

In practice, solving Einstein's equations is extraordinarily difficult: the metric components satisfy quasi-linear PDEs, not a simple local boundary-value problem. Moreover, $T^{\mu\nu}$ on the right–hand side is not known a priori (except in idealized cases such as the Poisson equation for electrostatics). One typically employs an initial-value (Cauchy) formulation, prescribing data on a three-dimensional hypersurface and evolving forward in time under the Einstein constraints (the Hamiltonian and momentum constraints). The rigorous existence and uniqueness of solutions (local in time) was proved by Y. Choquet-Bruhat and R. Geroch, leading to the so-called "Maximal Cauchy development" theorem. By adding four arbitrary functions ("gauge freedom") one fixes coordinates and obtains a well-posed evolution system [**?**].

It is also possible, by a theorem of Lovelock, to add to the left–hand side of (4.50) only a term proportional to $g_{\mu\nu}$ without violating the conservation law $\nabla^\mu G_{\mu\nu} = 0$. This yields the field equations with cosmological constant Λ:

$$R_{\mu\nu} - \tfrac{1}{2}\, g_{\mu\nu}\, R + \Lambda\, g_{\mu\nu} \;=\; 8\pi G\, T_{\mu\nu}. \tag{4.52}$$

The constant Λ was introduced by Einstein to obtain a static (non–expanding, non–contracting) universe. If $\Lambda > 0$, the equations admit a repulsive gravitational effect; observations since 1998 indicate a tiny positive value, $\Lambda \sim 10^{-52}\,\mathrm{m}^{-2}$. In modern cosmology the term Λ is often interpreted as vacuum energy, with $\rho_\Lambda = \Lambda/8\pi G$. In this work we shall, unless otherwise stated, set $\Lambda = 0$.

In *geometrized units* ($G = c = 1$), equation (4.50) becomes

$$G_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2}\, g_{\mu\nu} R = 8\pi\, T_{\mu\nu}, \tag{4.53}$$

and its trace–reversed form is obtained by contracting with $g^{\mu\nu}$:

$$R_{\mu\nu} - \tfrac{1}{2}\, g_{\mu\nu} R = 8\pi\, T_{\mu\nu} \quad \Longleftrightarrow \quad R_{\mu\nu} = 8\pi\Big(T_{\mu\nu} - \tfrac{1}{2}\, g_{\mu\nu} T\Big), \tag{4.54}$$

where $T = g^{\mu\nu} T_{\mu\nu}$ is the trace of the stress–energy tensor.

In vacuum ($T_{\mu\nu} = 0$), one then has

$$R_{\mu\nu} = 0, \tag{4.55}$$

so that the Einstein and Ricci tensors both vanish and the only nontrivial curvature is encoded in the Weyl tensor.

Although vacuum Einstein equations admit many solutions (gravitational waves, black holes, cosmological models), finding explicit, analytic forms is generally extremely challenging. One often resorts to perturbative or numerical methods to explore realistic scenarios such as binary mergers or cosmological evolution.

Chapter 5

Exact Solutions and Solution generating techniques

5.1 Weyl's class of stationary axisymmetric solutions

Very shortly after Einstein published his field equations for general relativity, Weyl (1917) and Levi–Civita (1918) considered static, axially symmetric solutions. In this chapter we will briefly review these "Weyl metrics" with emphasis on a few important special cases, particularly those which are asymptotically flat at large distances either along the axis or far from a finite number of sources. Despite their obvious and simple symmetry properties, the physical interpretation of these solutions is generally far from trivial. For example, one must be careful identifying "the axis" in coordinates that are periodic in the angular coordinate. Without such identification, a space–time with both spatial and temporal symmetries might be misinterpreted as plane-symmetric. Further, in the natural coordinates for the assumed symmetry, the behavior of any curvature singularities can depend sensitively on the direction from which they are approached. This subtlety occurs for singularities both on the axis and elsewhere.

A general line element possessing both a timelike Killing field ∂_t and an axial Killing field ∂_ϕ can be written in the canonical form

$$ds^2 = -e^{2U(\eta,\xi)}\,dt^2 + e^{-2U(\eta,\xi)}\Big[e^{2\gamma(\eta,\xi)}\big(d\eta^2 + d\xi^2\big) + \rho^2(\eta,\xi)\,d\phi^2\Big], \tag{5.1}$$

where U, γ, and ρ depend only on the coordinates (η, ξ), and the vacuum field equations (with $\Lambda = 0$) imply that

$$\frac{\partial^2 U}{\partial \eta^2} + \frac{1}{\rho}\frac{\partial \rho}{\partial \eta}\frac{\partial U}{\partial \eta} + \frac{\partial^2 U}{\partial \xi^2} = 0, \tag{5.2}$$

together with its ξ–η analogue.

Equation (5.2) is nothing but Laplace's equation $\nabla^2 U = 0$ for an axially symmetric potential U in an *auxiliary* Euclidean 3-space with "cylindrical" coordinates (ρ, ϕ, z), even though (η, ξ) here have a different meaning. In these "cylindrical coordinates" the line element becomes

$$ds^2 = -e^{2U(\rho,z)}\,dt^2 + e^{-2U(\rho,z)}\Big[e^{2\gamma(\rho,z)}\big(d\rho^2 + dz^2\big) + \rho^2\,d\phi^2\Big], \tag{5.3}$$

with U and γ functions of (ρ, z), $t \in (-\infty, \infty)$, $\rho \in [0, \infty)$, and $\phi \in [0, 2\pi)$. With $\Lambda = 0$, the vacuum field equation for U is simply

$$U_{,\rho\rho} + \frac{1}{\rho}\,U_{,\rho} + U_{,zz} = 0, \tag{5.4}$$

which one readily recognizes as Laplace's equation in Euclidean 3-space for an axially symmetric function $U(\rho, z)$. Since this is a linear equation, any superposition of harmonic functions U produces another solution. Once U is specified, the remaining metric function γ can be obtained by quadratures:

$$\gamma_{,\rho} = \rho\big(U_{,\rho}^2 + U_{,z}^2\big), \quad \gamma_{,z} = 2\,\rho\,U_{,\rho}\,U_{,z}. \tag{5.5}$$

These first-order equations are integrable by virtue of (5.4), so in principle all Weyl solutions can be written down formally.

Yet one should beware of singularities on the axis ($\rho = 0$) unless $\gamma \to 0$ as $\rho \to 0$. If that condition fails, the solution will have a 'conical' singularity, much like a misplaced plane sheet in space–time. In other words because U and γ can be chosen arbitrarily (subject only to (5.4)), the Weyl class describes all static, axisymmetric vacuum geometries. However, not every choice corresponds to a physically reasonable source; regularity on the axis and correct asymptotic behavior must be checked case by case.

5.1.1 Flat solutions within a Weyl metric

Even within the Weyl class, some choices of the harmonic potential U yield merely flat space–time, albeit in nontrivial coordinates.

(i) Trivial Minkowski. Choosing

$$U = 0, \qquad \gamma = 0, \tag{5.6}$$

in (5.3) gives

$$ds^2 = -dt^2 + d\rho^2 + dz^2 + \rho^2\, d\phi^2,$$

the usual Minkowski metric in cylindrical coordinates.

(ii) Uniformly Accelerated Frame. Let us choose

$$U = \ln \rho, \qquad \gamma = \ln \rho. \tag{5.7}$$

Then the line element becomes

$$ds^2 = -\rho^2\, dt^2 + d\rho^2 + dz^2 + \rho^2\, d\phi^2, \tag{5.8}$$

which is again flat, but now in Rindler-like (uniformly accelerated) coordinates. Indeed, by setting

$$\tilde{z} = z, \quad \tilde{\rho} = \rho, \quad t = \tau, \quad \phi = \varphi,$$

and introducing a new "vertical" coordinate $\tilde{z}$ via

$$z = \tfrac{1}{2}\big(\tilde{z}^2 - \tilde{\rho}^2\big),$$

one recovers the standard Rindler metric

$$ds^2 = -\tilde{z}^2\, d\tau^2 + d\tilde{z}^2 + d\tilde{\rho}^2 + \tilde{\rho}^2\, d\varphi^2.$$

The surface $\tilde{z} = 0$ corresponds to the acceleration horizon.

(iii) Gautreau–Hoffman "Semi-Infinite" Line Source. Gautreau and Hoffman (1969) found the unique nontrivial Weyl harmonic function leading to another flat–space representation,

$$U = \tfrac{1}{2}\ln\Big(\sqrt{\rho^2 + z^2} + z\Big), \qquad \gamma = \tfrac{1}{2}\ln\frac{\sqrt{\rho^2 + z^2} + z}{2\sqrt{\rho^2 + z^2}}. \tag{5.9}$$

This solution again has vanishing Riemann tensor, but can be interpreted as arising from a semi-infinite Newtonian line source of mass density $\sigma = \frac{1}{2}$ located on the negative z–axis. A change of coordinates shows it to be another uniformly accelerated frame, with the "source" now lying on the acceleration horizon.

Therefore three different choices of U here all give flat space–time—yet in wildly different guises: inertial, Rindler, and the Gautreau–Hoffman frame. Two of these ‘sources’ live on horizons and reflect the coordinate artifacts of acceleration rather than true gravitational fields. This teaches us a valuable lesson: the Newtonian potential analog U in an auxiliary 3-space does not by itself guarantee a physical mass distribution in the four-dimensional geometry and the full curvature and global structure have to be studied before assigning a physical source.

5.1.2 Weyl's solution

An important family of exact, asymptotically flat, static and axially symmetric vacuum solutions is obtained by passing from Weyl's "cylindrical" coordinates (ρ, z) to "spherical" coordinates

$$\rho = r\sin\theta, \quad z = r\cos\theta. \tag{5.10}$$

In these coordinates the line element (5.3) reads

$$ds^2 = -e^{2U(r,\theta)}\,dt^2 + e^{-2U(r,\theta)}\big[e^{2\gamma(r,\theta)}(dr^2 + r^2 d\theta^2) + r^2\sin^2\theta\,d\phi^2\big]. \tag{5.11}$$

The vacuum field equation (5.4) becomes

$$r^2\,U_{,rr} \;+\; 2r\,U_{,r} \;+\; U_{,\theta\theta} \;+\; \cot\theta\,U_{,\theta} \;=\; 0, \tag{5.12}$$

which is nothing other than the axisymmetric Laplace equation on flat three-space in standard spherical coordinates. The asymptotically flat, regular solutions of (5.12) admit an expansion in Legendre polynomials $P_n(\cos\theta)$:

$$U(r,\theta) = -\sum_{n=0}^{\infty} a_n\,r^{-(n+1)}\,P_n(\cos\theta). \tag{5.13}$$

Here the coefficients a_n play the rôle of "mass multipole moments" of the source. Once U is prescribed, the function γ follows from the quadrature

$$\gamma(r,\theta) = -\sum_{l,m=0}^{\infty} a_l\, a_m \,\frac{(l+1)(m+1)}{l+m+2}\,\frac{P_l(\cos\theta)P_m(\cos\theta) - P_{l+1}(\cos\theta)P_{m+1}(\cos\theta)}{r^{\,l+m+2}}. \tag{5.14}$$

These are the *Weyl solutions*, each term corresponding to the nonlinear gravitational effect of a Newtonian multipole. It's remarkably elegant that the entire infinite set of axisymmetric vacuum solutions reduces to choosing a sequence $\{a_n\}$ in a single harmonic expansion, just like in electrostatics. But be wary: the nonlinear "dressing" encoded in γ can introduce subtle singularities away from the axis unless the moments satisfy extra regularity conditions.

Levi–Civita's Special Case. Levi–Civita (1917) discovered a two-parameter subfamily of (5.11) characterized by three constants p_0, p_2, p_3 subject to

$$p_0 + p_2 + p_3 = 1, \quad p_0^2 + p_2^2 + p_3^2 = 1. \tag{5.15}$$

Introducing

$$\sigma = \tfrac{1}{2}(p_2 - p_3), \quad \Sigma = 1 - 2\sigma + 4\sigma^2, \tag{5.16}$$

one may set

$$p_0 = 2\sigma\Sigma^{-1},\ p_2 = 2\sigma(2\sigma-1)\Sigma^{-1},\ p_3 = (1-2\sigma)\Sigma^{-1}, \tag{5.17}$$

and rescale coordinates via

$$t \mapsto k^{-p_0}t,\ r \mapsto k^{p_2}r,\ z \mapsto k^{p_3}z,\ \phi \mapsto k^{-p_3}\phi, \tag{5.18}$$

introducing an overall constant k. The metric then becomes

$$ds^2 = -\rho^{4\sigma}\, dt^2 + k^2\, \rho^{4\sigma(2\sigma-1)}\big(d\rho^2 + dz^2\big) + \rho^{2(1-2\sigma)}\, d\phi^2, \tag{5.19}$$

where $\rho = r\sin\theta$, $z = r\cos\theta$. Comparing with (5.3), one reads off

$$U(\rho) = 2\sigma\ln\rho, \quad \gamma(\rho) = 4\sigma^2\ln\rho + \ln k. \tag{5.20}$$

The function $U = 2\sigma\ln\rho$ is just the Newtonian potential of an infinite line of mass density σ. Indeed, near $\rho = 0$ the metric (5.19) has a conical singularity whose deficit angle is proportional to σ. Israel (1977) showed that this "mass per unit length" interpretation is consistent only for $\sigma > 0$ and small; otherwise the pressures and red-shift blow up. Levi–Civita's metric thus emerges as the simplest nontrivial Weyl solution, and serves as a cautionary example: although its Newtonian analog is straightforward, the full four-dimensional geometry exhibits subtleties (singular axes, horizons, and "conical" stresses) that have no counterpart in classical Newtonian gravity.

5.1.3 Schwartzschild

The Schwarzschild metric is the unique, static, spherically symmetric vacuum solution, asymptotically flat at spatial infinity as we will shortly see. Its usual form is

$$ds^2 = -\Big(1 - \tfrac{2m}{r}\Big)dt^2 + \Big(1 - \tfrac{2m}{r}\Big)^{-1}dr^2 + r^2\big(d\theta^2 + \sin^2\theta\, d\phi^2\big).$$

To cast it into Weyl's form (5.3), one introduces new "prolate spheroidal" coordinates x, y via

$$r = m\,(x+1), \quad \cos\theta = y, \tag{5.21}$$

so that

$$ds^2 = -\frac{x-1}{x+1}\,dt^2 + m^2\frac{x+1}{x-1}\,dx^2 + m^2\frac{(x+1)^2}{1-y^2}\,dy^2 + m^2(x+1)^2(1-y^2)\,d\phi^2.$$

The static region $r > 2m$ corresponds to $x > 1$. Next define

$$\rho = m\sqrt{(x^2-1)(1-y^2)}, \quad z = m\,x\,y, \tag{5.22}$$

so that the metric takes the Weyl canonical form (5.3) with

$$e^{2U} = \frac{R_+ + R_- - 2m}{R_+ + R_- + 2m}, \quad e^{2\gamma} = \frac{(R_+ + R_-)^2 - 4m^2}{4\,R_+\,R_-}, \tag{5.23}$$

where

$$R_\pm = \sqrt{\rho^2 + (z \pm m)^2}.$$

Equivalently,

$$U = \tfrac{1}{2}\ln\!\Big(\frac{R_- + z - m}{R_+ + z + m}\Big), \tag{5.24}$$

which is formally identical to the Newtonian potential of a finite rod of length $2m$ on the z–axis, with uniform mass density $\sigma = \frac{1}{2}$. The static Schwarzschild region $r > 2m$ is thus described by a finite rod potential in Weyl coordinates. But beware: although U matches a Newtonian rod, the actual space–time has a horizon at $r = 2m$ (where the rod "ends"), beyond which these coordinates break down.

Mapping the coordinate segments onto the Weyl axis shows:

Schwarzschild region	$\leftrightarrow$	Weyl axis segment
Half–axis behind the hole ($\theta = \pi$, $r \geq 2m$)	$\leftrightarrow$	$\rho = 0,\ z \leq -m,$
Horizon ($r = 2m,\ 0 < \theta < \pi$)	$\leftrightarrow$	$\rho = 0,\ -m < z < m,$
Other half–axis ($\theta = 0,\ r \geq 2m$)	$\leftrightarrow$	$\rho = 0,\ z \geq +m.$

This mapping highlights that although the Newtonian analogy suggests a rod source, the actual geometry has a horizon in its middle. Moreover, interpreting Schwarzschild as a superposition of two semi-infinite rods—one of positive density $\sigma = +\frac{1}{2}$ on $z > m$ and another of negative density $\sigma = -\frac{1}{2}$ on $z < -m$—reveals that the global structure cannot be read off from U alone.

5.1.4 Zipoy–Voorhees

As we have seen, the Schwarzschild solution in Weyl form corresponds to the Newtonian potential of a finite rod of length 2ℓ with mass-per-unit-length $\sigma = \frac{1}{2}$. More generally, one may take any $\sigma \in \mathbb{R}$ and place a rod of length 2ℓ along the z-axis, $z \in (-\ell, \ell)$. The corresponding Weyl potential is

$$U(\rho, z) = \sigma \ \ln\frac{R_- + z - \ell}{R_+ + z + \ell}, \quad R_\pm = \sqrt{\rho^2 + (z \pm \ell)^2}\,. \tag{5.25}$$

By straightforward algebra the metric functions become

$$e^{2U} = \Big(\frac{R_+ + R_- - 2\ell}{R_+ + R_- + 2\ell}\Big)^{m/\ell}, \quad e^{2\gamma} = \Big(\frac{(R_+ + R_-)^2 - 4\ell^2}{4R_+R_-}\Big)^{m^2/\ell^2}, \tag{5.26}$$

where the rod's total mass is $m = 2\sigma\ell$. These two parameters (m, ℓ) (or equivalently (m, δ) with $\delta = m/\ell$) label the *Zipoy–Voorhees*, or γ–metric, which was first discovered by Bach and Weyl (1922) and Darmois (1927). Key special cases: - $\delta = 1$ $(\sigma = \frac{1}{2})$ reproduces Schwarzschild. - $\ell \to 0$ with m fixed gives the Curzon–Chazy solution (a point mass). - $\ell \to \infty$ with $\sigma = m/2\ell \to 0$ recovers the Levi–Civita metric (an infinite line).

Except for $\delta = 0, 1$, the segment $\rho = 0$, $-\ell < z < \ell$ is a true curvature singularity, not hidden by any horizon. Moreover, near $\rho \to 0$ one finds the induced circumference of a small circle,

$$C(\rho) \approx 2\pi\, \rho^{1-\delta}\, \ell^\delta \quad (\rho \ll \ell),$$

which diverges for $\delta > 1$ and vanishes for $0 < \delta < 1$. Kodama and Hikida (2003) have shown that for $\delta < 0$ the "rod" is ring-like, and for $0 < \delta < 1$ it is rod-like, both nakedly singular at the ends.

In prolate spheroidal coordinates

$$\rho = \ell\sqrt{(x^2-1)(1-y^2)}, \quad z = \ell\, x\, y,$$

the metric simplifies to

$$ds^2 = -e^{2U}dt^2 + \Sigma^2\Big(\frac{dx^2}{x^2-1} + \frac{dy^2}{1-y^2}\Big) + R^2\, d\phi^2, \tag{5.27}$$

with

$$e^{2U} = \Big(\frac{x-1}{x+1}\Big)^{\delta}, \quad e^{-2\gamma} = \Big(\frac{x^2-1}{x^2-y^2}\Big)^{\delta^2}, \tag{5.28}$$

$$\Sigma^2 = \ell^2\,(x+1)^{\,\delta+1}(x-1)^{\,\delta-1}(x^2-y^2)^{1-\delta^2}, \quad R^2 = \ell^2\,(x+1)^{\,1+\delta}(x-1)^{\,1-\delta}(1-y^2).$$

Here again one sees that unless $\delta = 0, 1$ the axis $x = 1$ (i.e. $\rho = 0$, $|y| < 1$) harbors a directional singularity.]

5.2 Ernst Potential

Motivation

Perturbations ("disturbances") of black holes and the construction of nearby exact or quasi-exact geometries are central in mathematical relativity and black-hole physics. Beyond linear perturbation theory, the Einstein equations admit powerful reductions under symmetry. In the stationary, axisymmetric, vacuum case, the *Ernst formalism* compresses the nonlinearity into a single complex scalar equation. This subsection derives the Ernst equation, shows how the Schwarzschild and Kerr solutions arise naturally within it, and outlines a perturbative viewpoint for stationary deformations.

Stationary Axisymmetry and the Papapetrou Form

Stationarity and axisymmetry provide two commuting Killing vectors, $\xi = \partial_t$ and $\eta = \partial_\phi$, with $[\xi, \eta] = 0$. In adapted Weyl–Papapetrou coordinates (t, ρ, z, ϕ) the metric may be written as

$$\mathrm{d}s^2 = -f(\rho,z)\big[\mathrm{d}t - \omega(\rho,z)\,\mathrm{d}\phi\big]^2 + f(\rho,z)^{-1}\Big(e^{2\gamma(\rho,z)}(\mathrm{d}\rho^2 + \mathrm{d}z^2) + \rho^2\,\mathrm{d}\phi^2\Big), \quad (5.29)$$

with $f > 0$. Let $f \equiv -\xi\cdot\xi$ be the norm of the stationary Killing field and define the *twist* one-form

$$\omega_\mu \equiv \epsilon_{\mu\nu\alpha\beta}\,\xi^\nu \nabla^\alpha \xi^\beta. \quad (5.30)$$

In vacuum, $\nabla_{[\mu}\omega_{\nu]} = 0$, so locally $\omega_\mu = \nabla_\mu \chi$ for a *twist potential* χ. The vacuum Einstein equations reduce to PDEs on the 3-space of Killing orbits, i.e. flat 3-space with cylindrical coordinates (ρ, z, ϕ), where fields are ϕ-independent. Using the flat-space gradient/divergence ∇ and dot product in this auxiliary space (and suppressing the trivial ϕ-direction), one standard form is

$$\nabla\cdot\big(\rho f^{-2}\,\nabla\chi\big) = 0, \quad (5.31)$$

$$\nabla\cdot\big(\rho\,\nabla f\big) = \rho f^{-1}\|\nabla f\|^2 - \rho f^{-3}\|\nabla\chi\|^2. \quad (5.32)$$

Once (f, χ) are known, γ follows by quadratures:

$$\gamma_{,\rho} = \frac{\rho}{4f^2}\Big[(f_{,\rho})^2 - (f_{,z})^2 + (\chi_{,\rho})^2 - (\chi_{,z})^2\Big], \quad (5.33)$$

$$\gamma_{,z} = \frac{\rho}{2f^2}\Big(f_{,\rho}f_{,z} + \chi_{,\rho}\chi_{,z}\Big). \quad (5.34)$$

The Complex Ernst Potential and the Ernst Equation

Define the complex Ernst potential

$$\mathcal{E}(\rho, z) \equiv f(\rho, z) + i\,\chi(\rho, z), \tag{5.35}$$

so that Eqs. (5.31)–(5.32) collapse to the single complex *Ernst equation*

$$(\mathrm{Re}\,\mathcal{E})\,\nabla^2\mathcal{E} = (\nabla\mathcal{E})\cdot(\nabla\mathcal{E}), \tag{5.36}$$

where ∇ and ∇^2 act with respect to the flat 3-metric $\mathrm{d}\ell^2 = \mathrm{d}\rho^2 + \mathrm{d}z^2 + \rho^2\mathrm{d}\phi^2$, and axisymmetry implies $\partial_\phi\mathcal{E} = 0$. Eq. (5.36) arises as the Euler–Lagrange equation of the variational principle

$$\delta\int \frac{\nabla\mathcal{E}\cdot\nabla\bar{\mathcal{E}}}{(\mathrm{Re}\,\mathcal{E})^2}\,\rho\,\mathrm{d}\rho\,\mathrm{d}z\,\mathrm{d}\phi = 0, \tag{5.37}$$

exhibiting a nonlinear sigma-model structure with target space $\mathrm{SU}(1,1)/\mathrm{U}(1)$. The metric function γ does not feed back into (5.36) and is obtained by integrating (5.33)–(5.34) once $\mathcal{E}$ is known. If $\chi \equiv 0$ (no twist), then $\mathcal{E}$ is real and (5.36) reduces to $\nabla^2(\ln f) = 0$. Writing $U \equiv \frac{1}{2}\ln f$, one obtains the Weyl form of the metric,

$$\mathrm{d}s^2 = -e^{2U}\,\mathrm{d}t^2 + e^{-2U}\Big(e^{2\gamma}(\mathrm{d}\rho^2 + \mathrm{d}z^2) + \rho^2\,\mathrm{d}\phi^2\Big), \tag{5.38}$$

with U *harmonic* in (ρ, z). This class contains all static, axisymmetric, vacuum solutions (Weyl solutions).

Coordinate Choices and Separability: Prolate Spheroidal Coordinates

It is often convenient to adopt prolate spheroidal coordinates (x, y) with $x \geq 1$ and $-1 \leq y \leq 1$ defined by

$$\rho = \kappa\sqrt{x^2-1}\,\sqrt{1-y^2}, \qquad z = \kappa\,xy, \qquad (\kappa > 0), \tag{5.39}$$

where the flat-space Laplacian separates as

$$\nabla^2\Phi = \frac{1}{\kappa^2(x^2-y^2)}\Big[\partial_x\big((x^2-1)\,\partial_x\Phi\big) + \partial_y\big((1-y^2)\,\partial_y\Phi\big)\Big]. \tag{5.40}$$

Axisymmetric harmonic functions then factorize in associated Legendre functions of x and y, a fact that underlies compact expressions for black-hole solutions in the Ernst picture.

Static Weyl Class and Schwarzschild via Ernst

In the static sector ($\chi = 0$), choose a harmonic potential $U(\rho, z)$; the function γ follows from Eqs. (5.33)–(5.34). A particularly compact representation of the Schwarzschild solution in prolate spheroidal coordinates is obtained by the *constant-phase* Ernst potential

$$\mathcal{E}(x,y) = \frac{x-1}{x+1}, \tag{5.41}$$

which is real (hence static). Identifying

$$r = M(x+1), \qquad y = \cos\theta, \tag{5.42}$$

reconstruction of the metric from $f = \operatorname{Re}\mathcal{E} = (x-1)/(x+1)$ and the quadratures for γ yields the standard Schwarzschild form

$$\mathrm{d}s^2 = -\Big(1-\frac{2M}{r}\Big)\mathrm{d}t^2 + \Big(1-\frac{2M}{r}\Big)^{-1}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2). \tag{5.43}$$

This example illustrates how spherical symmetry emerges as a special static Ernst configuration.

Kerr from the Ernst Potential

Introduce an auxiliary complex function $\Phi(x, y)$ and define

$$\mathcal{E} = \frac{\Phi - 1}{\Phi + 1}. \tag{5.44}$$

A separated solution that satisfies the Ernst equation in prolate coordinates is given by the linear combination

$$\Phi(x, y) = x\cos\alpha + i\, y\sin\alpha, \qquad 0 \le \alpha < \frac{\pi}{2}. \tag{5.45}$$

With the identifications

$$\kappa = \sqrt{M^2 - a^2}, \qquad \tan\alpha = \frac{a}{\kappa}, \qquad r = M + \kappa x, \qquad y = \cos\theta, \tag{5.46}$$

one reconstructs the Boyer–Lindquist form of the Kerr metric

$$\begin{aligned} \mathrm{d}s^2 = & -\Big(1 - \frac{2Mr}{\Sigma}\Big)\mathrm{d}t^2 - \frac{4Mar\sin^2\theta}{\Sigma}\,\mathrm{d}t\,\mathrm{d}\phi + \frac{\Sigma}{\Delta}\,\mathrm{d}r^2 + \Sigma\,\mathrm{d}\theta^2 \\ & + \Big(r^2 + a^2 + \frac{2Ma^2 r\sin^2\theta}{\Sigma}\Big)\sin^2\theta\,\mathrm{d}\phi^2, \end{aligned} \tag{5.47}$$

where $\Sigma = r^2 + a^2\cos^2\theta$ and $\Delta = r^2 - 2Mr + a^2$. In the Papapetrou language, $f = \mathrm{Re}\,\mathcal{E}$ and the twist potential $\chi = \mathrm{Im}\,\mathcal{E}$ determine the dragging function ω via first-order relations; γ is then found by quadratures (5.33)–(5.34) and the mapping (5.46).

Perturbative Viewpoint for "Disturbed" Stationary Black Holes

The variational structure of (5.37) enables controlled perturbation theory. Let

$$\mathcal{E} = \mathcal{E}_0 + \epsilon\,\mathcal{E}_1 + \epsilon^2\,\mathcal{E}_2 + \cdots, \qquad \mathcal{E}_0 = \frac{x - 1}{x + 1} \quad \text{(Schwarzschild)}. \tag{5.48}$$

Expanding (5.36) to $\mathcal{O}(\epsilon)$ gives the linear equation

$$\mathcal{L}[\mathcal{E}_1] \equiv E_0\,\nabla^2\mathcal{E}_1 - 2\,\nabla E_0\!\cdot\!\nabla\mathcal{E}_1 + (\mathrm{Re}\,\mathcal{E}_1)\,\nabla^2 E_0 = 0, \qquad (E_0 \equiv \mathcal{E}_0 \in \mathbb{R}). \tag{5.49}$$

Taking the imaginary part (which controls the first nonzero twist) yields a decoupled elliptic PDE

$$E_0\,\nabla^2(\mathrm{Im}\,\mathcal{E}_1) - 2\,\nabla E_0\!\cdot\!\nabla(\mathrm{Im}\,\mathcal{E}_1) = 0, \tag{5.50}$$

whose separated solutions on the prolate background are spanned by combinations $Q_\ell(x)P_\ell(y)$ (with Q_ℓ Legendre Q-functions). The exact Kerr family corresponds to a special nonperturbative choice resummed via the ansatz (5.45). More general stationary deformations (e.g. quadrupolar distortions or external fields) can be organized in this basis, with regularity and boundary conditions fixing the physical branch.

5.3 Solution-Generating Techniques from Ernst Equation

In stationary, axisymmetric *electrovacuum* General Relativity, the Einstein–Maxwell equations reduce, after 3D reduction on the stationary Killing orbits, to a nonlinear sigma model whose target admits a large isometry group. In the Ernst formulation, these isometries act as fractional-linear (Möbius-type) maps on the pair of complex potentials $(\mathcal{E}, \Phi)$ and generate new exact solutions from a known *seed*. We first derive the electrovac Ernst equations and the sigma-model structure, then we write the finite $SU(2,1)$ maps in explicit form (Harrison and Ehlers), and finally apply them to construct black holes immersed in a magnetic (Melvin) universe.

5.3.1 3D reduction, electrovac Ernst equations, and target-space geometry

Consider a stationary, axisymmetric spacetime with two commuting Killing fields $\xi = \partial_t$ and $\eta = \partial_\phi$. In Weyl–Papapetrou coordinates (t, ρ, z, ϕ) the metric and the Maxwell field are taken independent of t and ϕ:

$$\mathrm{d}s^2 = -f(\rho,z)\,[\mathrm{d}t - \omega(\rho,z)\,\mathrm{d}\phi]^2 + f(\rho,z)^{-1}\big(e^{2\gamma(\rho,z)}(\mathrm{d}\rho^2 + \mathrm{d}z^2) + \rho^2\mathrm{d}\phi^2\big)\,, \qquad f > 0, \tag{5.51}$$

and the field strength two-form F admits electric and magnetic scalar potentials on the 3-space of Killing orbits.

Electromagnetic potentials. Introduce the electric and magnetic potentials $v(\rho, z)$ and $u(\rho, z)$ via

$$E_a \equiv F_{a0} = \partial_a v, \qquad B^a \equiv \tfrac{1}{2}\,\epsilon^{abc}F_{bc} = \frac{1}{\rho}\,\epsilon^{ab}\partial_b u, \tag{5.52}$$

where indices $a, b \in \{\rho, z\}$, ϵ^{ab} is the Levi-Civita symbol on the flat (ρ, z)-plane, and all fields are axisymmetric ($\partial_\phi = 0$). Define the complex electromagnetic Ernst potential

$$\Phi \equiv v + iu. \tag{5.53}$$

Let $f \equiv -\xi\cdot\xi$ and define the vacuum twist one-form of ξ,

$$\omega_\mu \equiv \epsilon_{\mu\nu\alpha\beta}\,\xi^\nu\nabla^\alpha\xi^\beta, \tag{5.54}$$

which in electrovac satisfies

$$\nabla_{[\mu}\omega_{\nu]} = -2\,\epsilon_{\mu\nu\alpha\beta}\,\xi^\alpha F^{\beta\gamma}\xi^\delta F_{\gamma\delta}. \tag{5.55}$$

It is convenient to *improve* the twist by subtracting the electromagnetic contribution and define a scalar twist potential χ by

$$\omega_\mu - 2\,\mathrm{Im}\big(\bar{\Phi}\,\partial_\mu\Phi\big) = \partial_\mu\chi. \tag{5.56}$$

The *gravitational* Ernst potential is then

$$\mathcal{E} \equiv f - |\Phi|^2 + i\,\chi. \tag{5.57}$$

Electrovac Ernst equations. Using the 3D flat-space operators $\nabla = (\partial_\rho, \partial_z)$ and ∇^2 in cylindrical coordinates (ρ, z, ϕ) (with axial independence), the coupled Einstein–Maxwell equations are equivalent to the pair

$$\big(\Re\mathcal{E} - |\Phi|^2\big)\,\nabla^2\mathcal{E} = \big(\nabla\mathcal{E} - 2\,\bar{\Phi}\,\nabla\Phi\big)\cdot\nabla\mathcal{E}, \tag{5.58}$$

$$\big(\Re\mathcal{E} - |\Phi|^2\big)\,\nabla^2\Phi = \big(\nabla\mathcal{E} - 2\,\bar{\Phi}\,\nabla\Phi\big)\cdot\nabla\Phi, \tag{5.59}$$

with the metric function f recovered as

$$f = \Re\mathcal{E} - |\Phi|^2. \tag{5.60}$$

Once $(\mathcal{E}, \Phi)$ are known, the remaining functions follow by quadratures:

$$\partial_\rho\omega = -\frac{\rho}{f^2}\Big[\Im(\mathcal{E}_{,\rho}) + 2\,\Re(\Phi\,\bar{\Phi}_{,\rho})\Big], \qquad \partial_z\omega = -\frac{\rho}{f^2}\Big[\Im(\mathcal{E}_{,z}) + 2\,\Re(\Phi\,\bar{\Phi}_{,z})\Big], \tag{5.61}$$

$$\gamma_{,\rho} = \frac{\rho}{4f^2}\big(f_{,\rho}^2 - f_{,z}^2 + \chi_{,\rho}^2 - \chi_{,z}^2 + 4\,|\Phi_{,\rho}|^2 - 4\,|\Phi_{,z}|^2\big)\,, \quad \gamma_{,z} = \frac{\rho}{2f^2}\big(f_{,\rho}f_{,z} + \chi_{,\rho}\chi_{,z} + 4\,\Re(\Phi_{,\rho}\bar{\Phi}_{,z})\big)\,. \tag{5.62}$$

Sigma-model structure and hidden symmetry. Equations (5.58)–(5.59) arise from the 3D action

$$S = \int \rho \, \mathrm{d}\rho \, \mathrm{d}z \, \mathrm{d}\phi \; \frac{\nabla\mathcal{E}\cdot\nabla\bar{\mathcal{E}} - 2\,(\bar{\Phi}\,\nabla\mathcal{E} - \nabla\Phi)\cdot(\Phi\,\nabla\bar{\mathcal{E}} - \nabla\bar{\Phi})}{(\Re\mathcal{E} - |\Phi|^2)^2}, \tag{5.63}$$

which is a nonlinear sigma model with target the Kähler coset $SU(2,1)/S(U(2) \times U(1))$. The target-space isometries $SU(2,1)$ act as *solution-generating* maps on $(\mathcal{E}, \Phi)$ and preserve (5.58)–(5.59). In vacuum ($\Phi \equiv 0$) the symmetry reduces to $SL(2,\mathbb{R})$ acting by Möbius maps on $\mathcal{E}$.[1]

5.3.2 Finite $SU(2,1)$ maps: Harrison (charging/magnetizing) and Ehlers (twist/NUT)

A convenient way to present the finite maps is via their direct action on $(\mathcal{E}, \Phi)$. Let $(\mathcal{E}_0, \Phi_0)$ be a seed satisfying (5.58)–(5.59) and define Λ-factors as indicated below.

Magnetic Harrison transformation (Melvin embedding). Given a real parameter B (magnetic field strength at infinity) define

$$\Lambda \equiv 1 + \tfrac{1}{2}B\,\Phi_0 - \tfrac{1}{4}B^2\,\mathcal{E}_0. \tag{5.64}$$

Then

$$\boxed{\mathcal{E} = \frac{\mathcal{E}_0}{|\Lambda|^2}, \qquad \Phi = \frac{\Phi_0 - \frac{1}{2}B\,\mathcal{E}_0}{\Lambda}, \qquad f = \frac{f_0}{|\Lambda|^2}.} \tag{5.65}$$

This immerses the seed into the Bonnor–Melvin magnetic universe; asymptotics are Melvin rather than flat.[3, 4, 7, 8]

[1] For derivations and the identification of the symmetry groups, see the original papers by Ernst and the Kinnersley–Ehlers program. [1, 2, 5, 6, 10]

Electric Harrison transformation (charging). For real q,

$$\Lambda_{\rm el} \equiv 1 - q\,\bar{\Phi}_0 - \tfrac{1}{2}q^2\,\bar{\mathcal{E}}_0, \qquad \boxed{\mathcal{E} = \frac{\mathcal{E}_0}{\Lambda_{\rm el}\bar{\Lambda}_{\rm el}}, \qquad \Phi = \frac{\Phi_0 + \frac{1}{2}q\,\mathcal{E}_0}{\Lambda_{\rm el}}, \qquad f = \frac{f_0}{\Lambda_{\rm el}\bar{\Lambda}_{\rm el}}.} \tag{5.66}$$

This adds net electric charge to a vacuum or neutral electrovac seed (e.g. Schwarzschild $\to$ Reissner–Nordström; Kerr $\to$ Kerr–Newman).[6, 5]

Ehlers transformation (vacuum $SL(2,\mathbb{R})$ and its electrovac embedding). In *vacuum* ($\Phi_0 \equiv 0$), a one-parameter representative with $c \in \mathbb{R}$ acts by a Möbius map on the Ernst potential:

$$\boxed{\mathcal{E} = \frac{\mathcal{E}_0}{1 + ic\,\mathcal{E}_0}, \quad (\Phi \equiv 0).} \tag{5.67}$$

It generates nonzero twist from a static seed and thus adds NUT-type "dual mass". In full electrovac, Ehlers extends to a coupled map on $(\mathcal{E}, \Phi)$ and is often composed with a duality rotation to keep the electromagnetic field aligned; explicit forms follow from the $SU(2,1)$ action.[9, 10]

Why these maps preserve the equations. Let $\mathcal{T} \in SU(2,1)$ act on the coset representative $\mathsf{M}(\mathcal{E}, \Phi)$ by $\mathsf{M} \mapsto \mathcal{T}^\dagger \mathsf{M} \mathcal{T}$; the sigma-model Lagrangian is $L = \frac{1}{2}\mathrm{Tr}[(\nabla \mathsf{M})\mathsf{M}^{-1}]^2$ which is manifestly invariant. Reading off the transformed $(\mathcal{E}, \Phi)$ from M' reproduces (5.65)–(5.67). (This avoids solving PDEs anew and is the conceptual reason the algebraic maps work.)[5, 6]

5.3.3 Black Holes in a Magnetic Universe

We now magnetize standard seeds using (5.65). The reconstruction of ω and γ uses (5.61)–(5.62). A useful rule of thumb: f always rescales as $f = f_0/|\Lambda|^2$, while the azimuthal coefficient acquires the inverse factor: $g_{\phi\phi} \sim \rho^2 f^{-1} \sim (\rho^2/f_0)\,|\Lambda|^2$; keeping these paired factors consistent prevents most algebraic slips.

Magnetized Schwarzschild (Ernst metric)

Take the static seed

$$\mathcal{E}_0 = 1-\frac{2M}{r}, \qquad \Phi_0 = 0, \qquad \mathrm{d}s_0^2 = -\Big(1-\frac{2M}{r}\Big)\mathrm{d}t^2+\Big(1-\frac{2M}{r}\Big)^{-1}\mathrm{d}r^2+r^2(\mathrm{d}\theta^2+\sin^2\theta\,\mathrm{d}\phi^2). \tag{5.68}$$

With $B \in \mathbb{R}$ and $\Lambda = 1-\frac{1}{4}B^2\mathcal{E}_0 = 1+\frac{1}{4}B^2r^2\sin^2\theta$, the magnetic Harrison map (5.65) yields

$$\boxed{\begin{aligned} &\mathrm{d}s^2 = \Lambda^2\Big[-\Big(1-\frac{2M}{r}\Big)\mathrm{d}t^2 + \Big(1-\frac{2M}{r}\Big)^{-1}\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2\Big] + \frac{r^2\sin^2\theta}{\Lambda^2}\,\mathrm{d}\phi^2, \\ &A_\phi = \frac{B}{2}\,\frac{r^2\sin^2\theta}{\Lambda} \qquad (\text{gauge } A_t = 0), \qquad \omega = 0, \end{aligned}} \tag{5.69}$$

the classic Ernst spacetime describing a Schwarzschild black hole immersed in Melvin's magnetic universe.[3, 8] At large r the metric approaches the Melvin flux tube; the axis $\sin\theta = 0$ is regular provided ϕ is rescaled if needed to remove conical deficit.[3]

Magnetized Kerr and Kerr–Newman (Ernst–Wild family)

Start from Kerr–Newman in Boyer–Lindquist form with parameters (M, a, Q); its electrovac Ernst potentials in prolate (or spheroidal) coordinates can be written compactly. Acting with (5.65) produces the Ernst–Wild magnetized black holes:[4, 7]

$$\mathcal{E} = \frac{\mathcal{E}_0}{|\Lambda|^2}, \qquad \Phi = \frac{\Phi_0 - \frac{1}{2}B\mathcal{E}_0}{\Lambda}, \qquad \Lambda = 1 + \tfrac{1}{2}B\Phi_0 - \tfrac{1}{4}B^2\mathcal{E}_0. \tag{5.70}$$

Rotation and charge mix nontrivially with B, deforming the ergoregion and the near-horizon geometry while preserving a smooth Killing horizon. Thermodynamic relations (first law/Smarr) require definitions adapted to Melvin asymptotics and have been clarified in modern work.[7]

5.3.4 Harrison transformations

We sketch the derivation of (5.65)–(5.66) from the $SU(2,1)$ action. A convenient coset representative $\mathsf{M}(\mathcal{E},\Phi)$ is

$$\mathsf{M} = \frac{1}{\Re\mathcal{E} - |\Phi|^2}\begin{pmatrix} 1 & -\Phi & -\bar{\mathcal{E}} \\ -\bar{\Phi} & |\Phi|^2 & \bar{\Phi}\,\bar{\mathcal{E}} \\ -\mathcal{E} & \Phi\,\mathcal{E} & |\mathcal{E}|^2 \end{pmatrix}, \qquad \mathsf{M}^\dagger = \mathsf{M}, \qquad \det\mathsf{M} = 1, \tag{5.71}$$

on which $SU(2,1)$ acts by $\mathsf{M} \mapsto \mathcal{T}^\dagger\mathsf{M}\mathcal{T}$. Choosing

$$\mathcal{T}_{\rm mag}(B) = \begin{pmatrix} 1 & \frac{B}{2} & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{B}{2} & 1 \end{pmatrix}, \qquad \mathcal{T}_{\rm el}(q) = \begin{pmatrix} 1 & 0 & 0 \\ -q & 1 & 0 \\ -\frac{q^2}{2} & q & 1 \end{pmatrix}, \tag{5.72}$$

one finds after algebra the fractional-linear actions

$$\text{Magnetic:}\quad \Lambda = 1 + \tfrac{1}{2}B\,\Phi_0 - \tfrac{1}{4}B^2\mathcal{E}_0, \qquad \mathcal{E} = \frac{\mathcal{E}_0}{|\Lambda|^2}, \qquad \Phi = \frac{\Phi_0 - \frac{1}{2}B\,\mathcal{E}_0}{\Lambda}, \tag{5.73}$$

$$\text{Electric:}\quad \Lambda_{\rm el} = 1 - q\,\bar{\Phi}_0 - \tfrac{1}{2}q^2\bar{\mathcal{E}}_0, \qquad \mathcal{E} = \frac{\mathcal{E}_0}{\Lambda_{\rm el}\bar{\Lambda}_{\rm el}}, \qquad \Phi = \frac{\Phi_0 + \frac{1}{2}q\,\mathcal{E}_0}{\Lambda_{\rm el}}, \tag{5.74}$$

hence (5.65)–(5.66). The invariance of S under $\mathsf{M} \mapsto \mathcal{T}^\dagger\mathsf{M}\mathcal{T}$ ensures the transformed pair solves (5.58)–(5.59).[5, 6]

5.3.5 Ehlers transformation: twist/NUT from a static seed

In vacuum ($\Phi \equiv 0$), the sigma model reduces to target $SL(2,\mathbb{R})/SO(2)$ with Ernst potential $\mathcal{E} = f + i\chi$. The finite Ehlers map (5.67) is generated by

$$\mathcal{S}(c) = \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \in SL(2,\mathbb{R}), \qquad \mathcal{E} \mapsto \frac{\mathcal{E}_0}{1 + ic\,\mathcal{E}_0}. \tag{5.75}$$

Effect on parameters near infinity. For an asymptotically flat static seed with

$$\mathcal{E}_0 = 1 - \frac{2M}{r} + \mathcal{O}(r^{-2}), \qquad \chi_0 = 0, \tag{5.76}$$

expand (5.67):

$$\mathcal{E} = \frac{1 - \frac{2M}{r} + \cdots}{1 + ic - ic\,\frac{2M}{r} + \cdots} = (1-ic)\left(1 - \frac{2M}{r}\right)\left[1 + ic + \frac{2icM}{r} + \cdots\right] = 1 - \frac{2M}{r} + i\,\frac{2cM}{r} + \cdots . \tag{5.77}$$

Thus the transformed solution acquires a NUT charge $N = c\,M$ through the r^{-1} term of $\Im\mathcal{E}$ (the twist). The resulting spacetime is Taub–NUT–type (or its accelerating analogue if the seed was accelerating).[9] For electrovac seeds ($\Phi_0 \neq 0$) one combines the Ehlers map with a $U(1)$ electromagnetic duality rotation so that electric/magnetic components mix consistently in the new frame.[10]

5.4 Time-Dependent Solutions: Canonical Families and Explicit Derivations

This section collects and derives representative *time-dependent* exact solutions of Einstein's equations, both cosmological and radiative. We treat: homogeneous/isotropic cosmologies (FLRW), homogeneous but anisotropic vacuum (Kasner), spherically symmetric inhomogeneous dust (LTB), fully inhomogeneous dust without Killing vectors (Szekeres), radiating null dust (Vaidya), exact plane gravitational waves (pp-waves), expanding algebraically special waves (Robinson–Trautman), and a compact mass embedded in an expanding universe (McVittie). Each subsection states the metric, computes the reduced Einstein equations, and integrates to the standard form, highlighting the minimal set of nontrivial curvature components that control the dynamics.

Throughout, we set $c = G = 1$. Our sign convention is $(-,+,+,+)$ and Einstein's equations read

$$G_{\mu\nu} \equiv R_{\mu\nu} - \tfrac{1}{2} R\, g_{\mu\nu} = 8\pi T_{\mu\nu}. \tag{5.78}$$

5.4.1 FLRW Cosmologies (perfect fluid)

Homogeneity and isotropy imply constant-curvature spatial slices. In comoving cosmic time,

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a(t)^2\, \gamma_{ij}\, \mathrm{d}x^i \mathrm{d}x^j, \qquad \gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j = \frac{\mathrm{d}r^2}{1-kr^2} + r^2(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\phi^2), \tag{5.79}$$

with $k \in \{0, \pm 1\}$. Stress–energy is a perfect fluid:

$$T_{\mu\nu} = (\rho + p) u_\mu u_\nu + p\, g_{\mu\nu}, \qquad u^\mu = \delta^\mu{}_t. \tag{5.80}$$

Nonzero Christoffel symbols are

$$\Gamma^t{}_{ij} = a\dot{a}\, \gamma_{ij}, \qquad \Gamma^i{}_{tj} = \frac{\dot{a}}{a}\, \delta^i{}_j, \qquad \Gamma^i{}_{jk}(\gamma) = {}^{(3)}\Gamma^i{}_{jk}. \tag{5.81}$$

Useful Ricci components:

$$R_{tt} = -3\, \frac{\ddot{a}}{a}, \tag{5.82}$$

$$R_{ij} = \Big(a\ddot{a} + 2\dot{a}^2 + 2k \Big)\, \gamma_{ij}. \tag{5.83}$$

The scalar curvature is

$$R = 6 \left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} \right]. \tag{5.84}$$

The nontrivial components G_{tt} and (any) G_{ii} yield

$$3\left(\frac{\dot{a}}{a}\right)^2 + 3\frac{k}{a^2} = 8\pi\rho, \tag{5.85}$$

$$-2\frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2 - \frac{k}{a^2} = 8\pi p. \tag{5.86}$$

Energy conservation $\nabla_\mu T^{\mu\nu} = 0$ gives

$$\dot{\rho} + 3H(\rho + p) = 0, \qquad H \equiv \dot{a}/a. \tag{5.87}$$

From (5.85)–(5.86):

$$\frac{\ddot{a}}{a} = -\frac{4\pi}{3}(\rho + 3p). \tag{5.88}$$

With an equation of state $p = w\rho$ ($w = \text{const}$), integration gives

$$\rho(a) = \rho_0\, a^{-3(1+w)}, \qquad a(t) \propto \begin{cases} t^{\frac{2}{3(1+w)}} & (k = 0,\ \Lambda = 0), \\ \sinh^{\frac{2}{3(1+w)}}\left(\frac{3}{2}(1+w)\sqrt{\frac{\Lambda}{3}}\,t\right) & (k = 0,\ \Lambda > 0), \end{cases} \tag{5.89}$$

with the usual special cases: dust ($w = 0 \Rightarrow a \propto t^{2/3}$), radiation ($w = 1/3 \Rightarrow a \propto t^{1/2}$).

5.4.2 Kasner Vacuum (Bianchi I)

Ansatz. Homogeneous, anisotropic, spatially flat vacuum:

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + t^{2p_1}\,\mathrm{d}x^2 + t^{2p_2}\,\mathrm{d}y^2 + t^{2p_3}\,\mathrm{d}z^2, \tag{5.90}$$

with constants (p_1, p_2, p_3) and nonzero Christoffels:

$$\Gamma^x{}_{xt} = \frac{p_1}{t}, \quad \Gamma^y{}_{yt} = \frac{p_2}{t}, \quad \Gamma^z{}_{zt} = \frac{p_3}{t}. \tag{5.91}$$

Ricci components evaluate to

$$R_{tt} = \sum_i p_i(1-p_i)\,t^{-2}, \tag{5.92}$$

$$R_{xx} = t^{2p_1}\left[p_1\Big(\sum_j p_j - 1\Big)\right]t^{-2}, \quad \text{(cyclic in } x, y, z\text{)}. \tag{5.93}$$

Vacuum $R_{\mu\nu} = 0$ gives the *Kasner constraints*:

$$\boxed{\sum_{i=1}^{3} p_i = 1, \qquad \sum_{i=1}^{3} p_i^2 = 1.} \tag{5.94}$$

A one-parameter family (up to permutation) is $p_1 = \frac{-u}{1+u+u^2}$, $p_2 = \frac{1+u}{1+u+u^2}$, $p_3 = \frac{u(1+u)}{1+u+u^2}$, $u \geq 1$.

5.4.3 Lemaître–Tolman–Bondi (LTB) Dust

Spherical symmetry, inhomogeneous *dust* ($p = 0$), comoving proper time t:

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \frac{\big(R'(t,r)\big)^2}{1+2E(r)}\,\mathrm{d}r^2 + R(t,r)^2\,\mathrm{d}\Omega^2, \qquad T_{\mu\nu} = \rho\,u_\mu u_\nu,\ u^\mu = \delta^\mu{}_t, \tag{5.95}$$

where prime $'$ is ∂_r, dot is ∂_t, and $E(r)$ is an arbitrary "energy" function.

$$\partial_t\left(\frac{R'}{\sqrt{1+2E}}\right) - \partial_r\left(\frac{\dot{R}}{\sqrt{1+2E}}\right) = 0 \;\Rightarrow\; \exists\, M(r):\quad \dot{R}^2 = \frac{2M(r)}{R} + 2E(r). \tag{5.96}$$

The $G^t{}_t = -8\pi\rho$ yields the density

$$\boxed{8\pi\rho(t,r) = \frac{2M'(r)}{R^2 R'}.} \tag{5.97}$$

Thus LTB reduces to quadrature of (5.96) given $E(r), M(r)$, with three kinematic classes:

$$E > 0: \text{ hyperbolic}, \qquad E = 0: \text{ parabolic}, \qquad E < 0: \text{ elliptic}. \tag{5.98}$$

The solution can be written parametrically (e.g. for $E < 0$ with η parameter), reproducing FLRW in the special case $M \propto r^3$, $E \propto -kr^2$.

5.4.4 Szekeres

The Szekeres metrics generalize LTB by dropping spherical symmetry. In *quasi-spherical* class:

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \frac{(\Phi_{,r} - \Phi\,\mathcal{E}_{,r}/\mathcal{E})^2}{\epsilon - k(r)}\,\mathrm{d}r^2 + \frac{\Phi(t,r)^2}{\mathcal{E}(r,x,y)^2}\big(\mathrm{d}x^2 + \mathrm{d}y^2\big), \tag{5.99}$$

where

$$\mathcal{E}(r,x,y) = \frac{1}{2S(r)}\Big[(x - P(r))^2 + (y - Q(r))^2\Big] + \frac{\epsilon}{2}, \qquad \epsilon = +1, 0, -1. \tag{5.100}$$

Dust stress–energy as in LTB: $T_{\mu\nu} = \rho\, u_\mu u_\nu$, $u^\mu = \delta^\mu{}_t$. Einstein's equations reduce to

$$\dot{\Phi}^2 = \frac{2M(r)}{\Phi} - k(r) + \frac{1}{3}\Lambda\,\Phi^2, \qquad 8\pi\rho = \frac{2\big(M' - 3M\,\mathcal{E}_{,r}/\mathcal{E}\big)}{\Phi^2\big(\Phi_{,r} - \Phi\,\mathcal{E}_{,r}/\mathcal{E}\big)}. \tag{5.101}$$

For $\mathcal{E}$ independent of x, y the metric reduces to LTB. In generic Szekeres, there are no Killing vectors; inhomogeneities enter via the functions (S, P, Q, k, M).

5.4.5 Vaidya

The Vaidya spacetime describes a spherically symmetric body that absorbs or emits *null radiation* ("null dust"), so that its mass varies along ingoing or outgoing lightlike directions. In advanced Eddington–Finkelstein coordinates (v, r, θ, ϕ) one has

$$\mathrm{d}s^2 = -\left(1 - \tfrac{2m(v)}{r}\right)\mathrm{d}v^2 + 2\,\mathrm{d}v\,\mathrm{d}r + r^2\,\mathrm{d}\Omega^2, \qquad \mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\!\theta\,\mathrm{d}\phi^2, \tag{5.102}$$

while in retarded coordinates (u, r, θ, ϕ) the outgoing version reads

$$\mathrm{d}s^2 = -\left(1 - \tfrac{2m(u)}{r}\right)\mathrm{d}u^2 - 2\,\mathrm{d}u\,\mathrm{d}r + r^2\,\mathrm{d}\Omega^2. \tag{5.103}$$

Substituting the advanced form into Einstein's equations yields a single nonvanishing mixed component of the Einstein tensor,

$$G_{vv} = \frac{2\,\dot{m}(v)}{r^2}, \qquad (\dot{} \equiv \tfrac{\mathrm{d}}{\mathrm{d}v}), \tag{5.104}$$

so the source is a pure null flux with stress–energy

$$T_{\mu\nu} = \frac{\dot{m}(v)}{4\pi r^2}\,k_\mu k_\nu, \qquad k_\mu \mathrm{d}x^\mu = \mathrm{d}v, \quad k^\mu k_\mu = 0, \tag{5.105}$$

which is traceless ($T^\mu{}_\mu = 0$) and satisfies the null/weak energy conditions when $\dot{m}(v) \geq 0$ (ingoing accretion). For the outgoing chart one has $T_{uu} = -\dot{m}(u)/(4\pi r^2)$, so energy conditions require $\dot{m}(u) \leq 0$ (radiation loss). When $m = \text{const}$ the metric reduces to Schwarzschild; for $\dot{m} \neq 0$ the spacetime is genuinely dynamical. The (future, outer) marginally trapped 2–spheres lie at the *apparent horizon*

$$r_{\mathrm{AH}}(v) = 2\,m(v), \tag{5.106}$$

which evolves according to the luminosity $\dot{m}$, while the event horizon is a global null surface that generally lags or leads r_{AH} depending on whether the black hole is growing or shrinking. The Misner–Sharp mass equals $m(v)$, so a quasi–local mass is built in. Radial null expansions for an outgoing/ingoing pair (ℓ^μ, n^μ) with $\ell \cdot n = -1$

are

$$\theta_{(\ell)} = \frac{r - 2m(v)}{r^2}, \qquad \theta_{(n)} = -\frac{2}{r}, \tag{5.107}$$

making the trapping structure explicit. The spacetime is algebraically special (type D) and admits a principal null congruence aligned with the radiation flow; in Newman–Penrose notation the only nonzero Ricci scalar is Φ_{00} for the ingoing case (or Φ_{22} for the outgoing case), while the instantaneous Weyl curvature matches Schwarzschild with

$$\Psi_2 = -\frac{m(v)}{r^3}. \tag{5.108}$$

Curvature invariants keep their leading Schwarzschild falloff (e.g. the Kretschmann scalar behaves like $K \sim m(v)^2/r^6$ near $r = 0$), signaling the physical singularity at $r = 0$ while remaining regular across the (dynamical) horizon in these coordinates. Physically, the Vaidya family models null–shell accretion/evaporation, gravitational collapse with radiative stages, and piecewise–constant mass profiles (thin null shells) matched to Schwarzschild across null hypersurfaces via Barrabès–Israel junction conditions; in the outgoing case, $m(u)$ coincides with the Bondi mass at $\mathscr{I}^+$ and $-\dot{m}(u)$ is the total luminosity carried by the radiation. Despite spherical symmetry (and hence no gravitational–wave degrees of freedom), Vaidya provides a canonical laboratory for *dynamical horizons*, tests of cosmic censorship (certain self–similar $m(v)$ profiles can yield locally naked singularities), and semiclassical backreaction models where $m(u)$ captures effective Hawking–like outflow.

Bibliography

[1] F. J. Ernst, "New Formulation of the Axially Symmetric Gravitational Field," *Phys. Rev.* **167**, 1175 (1968). (Ernst equation; variational form).

[2] F. J. Ernst, "New Formulation of the Axially Symmetric Gravitational Field. II," *Phys. Rev.* **168**, 1415 (1968). (Einstein–Maxwell version).

[3] F. J. Ernst, "Black holes in a magnetic universe," *J. Math. Phys.* **17**, 54 (1976). (Harrison map; Melvin embedding).

[4] F. J. Ernst and W. J. Wild, "Kerr black holes in a magnetic universe," *J. Math. Phys.* **17**, 182 (1976).

[5] H. Stephani et al., *Exact Solutions of Einstein's Field Equations*, 2nd ed., Cambridge Univ. Press (2003), Chapter "Special methods" (pp. 518–552). ($SU(2,1)$ symmetry; catalog of maps).

[6] (Reviews on) Kinnersley's $SU(2,1)$ symmetry and explicit transformation formulas for stationary Einstein–Maxwell fields.

[7] M. Astorino, "Mass of Kerr–Newman black holes in an external magnetic field," *Phys. Rev. D* **94**, 024019 (2016). (Thermodynamics in Melvin background).

[8] F. Hejda, "Black Holes and Magnetic Fields," WDS Proc. (2014). (Concise derivation of Melvin and Harrison action on Minkowski/Schwarzschild).

[9] J. Barrientos *et al.*, "Ehlers transformations as a tool for constructing accelerating NUT spacetimes," *Phys. Rev. D* **108**, 024059 (2023).

[10] J. Barrientos *et al.*, "Ernst equations and the $SU(2,1)$ symmetry" (2024). (Appendix with user-friendly rederivation; careful sign conventions).

Chapter 6

The Vanishing of TLNs of Kerr BHs and the role of symmetries

Note: The following chapter is written by me and my supervisor A.Kehagias as well as my supervisors colleague A.Riotto and has been published in JCAP.

6.1 Introduction

Gravitational Waves (GWs) and Black Holes (BHs) are key predictions of General Relativity (GR), validated by groundbreaking observations such as the detection of GWs from BH mergers by the LIGO and Virgo collaborations [1]. These detections have provided critical evidence supporting Einstein theory of gravity, showing no evidence for deviations from it [35].

During the inspiral phase of a compact binary system, such as those involving neutron stars or BHs, tidal interactions become significant when the orbital separation is sufficiently small. These tidal effects influence both the system's dynamics and the emitted GWs. The interplay between GWs and tidal effects is essential for refining binary inspiral models and testing GR under extreme conditions.

Tidal effects are characterized by parameters known as Love Numbers, which quantify an object's deformation in response to the gravitational field of its companion. In particular the static Tidal Love Numbers (TLNs) depend on the internal structure and composition of the compact objects undergoing tidal deformation [42]. These parameters play a pivotal role in modifying the gravitational waveform, with their contributions emerging at the fifth post-Newtonian order [20]. For example, the nonzero TLNs of neutron stars provide valuable insights into the equation of state of dense nuclear matter. In contrast, BHs are expected to have zero TLNs due to their lack of a rigid structure. This result is typically demonstrated using perturbation theory, showing that a linear tidal deformation with amplitude proportional to r^ℓ does not elicit an $r^{-\ell-1}$ response (ℓ being the corresponding multipole), resulting in vanishing static TLNs. Linear perturbations induced by external tidal forces cannot produce nonzero TLNs [5, 16, 15, 39, 38, 43, 36, 12, 37, 41, 33]. This phenomenon appears to stem from underlying hidden symmetries [25, 10, 9, 26, 27, 11, 28, 30, 6, 32, 3, 4, 17, 45].

Recent analyses have confirmed that static TLNs also vanish for second-order perturbations in the external tidal field [47, 46]. Furthermore, for the Schwarzschild BH, the vanishing of TLNs has been proven to hold for the parity-even perturbations at all orders in the external tidal field [34, 13].

The fact that the static TLN for BHs vanishes or not is of primary importance to distinguish BH mergers from neutron star mergers [14], having neutron stars a sizeable TLN. Furthermore, even the the merger of two spinless BHs give rise to a spinning Kerr BH. This calls for a unavoidable question: do Kerr BHs have a vanishing static TLN at any order in the external tidal force?

The case of rotating BHs, modeled by the Kerr solution, presents additional challenges. Rotation introduces frame-dragging effects and modifies the geometry of the spacetime, complicating the analysis of tidal interactions. Understanding the tidal response of Kerr BHs is essential, not only for theoretical completeness, but also for modeling gravitational waveforms from realistic astrophysical systems, where BHs are often expected to spin.

In this paper, we address this question of the vanishing of the static TLN of Kerr BHs by employing the Ernst formalism [18] and Weyl coordinates to analyze the tidal response of Kerr BHs. The Ernst potential provides a powerful framework for describing axially symmetric spacetimes, allowing us to incorporate rotation and non-linear effects systematically. By expressing the Kerr metric in prolate spheroidal coordinates, we generalize previous results for Schwarzschild BHs and demonstrate that the static tidal Love numbers of Kerr BHs vanish at all orders in the external tidal field. We will also identify the non-linear symmetries responsible for such a result.

This result highlights the robustness of the symmetry-based arguments that govern BH responses and underscores the distinctive nature of BHs as solutions to GR. The vanishing TLNs reaffirm the principle that BHs, unlike other compact objects, do not retain any permanent deformation under static tidal forces. This study contributes to the broader understanding of BH physics, offering new perspectives on their interaction with external fields and implications for gravitational wave astronomy.

We should stress however, that unlike the mass and spin of black holes, which are well-defined conserved charges and gauge-invariant quantities, the tidal Love numbers exhibit a different nature. While their definition is straightforward within Newtonian theory [24], the static TLN is neither a conserved charge nor a gauge-invariant quantity in general relativity. This has been discussed extensively in the literature (see, for instance, [23, 5, 29, 31]).

This ambiguity has led to an alternative approach: defining the linear static TLN as a Wilson coefficient obtained by matching an operator in the worldline effective action. However, carrying out this matching process necessitates a specific gauge choice—typically the de Donder gauge—to simplify the graviton propagator. Consequently, translating results from the de Donder gauge to another gauge, such as the Regge-Wheeler (RW) gauge used at the linear level, becomes necessary for the matching procedure.

A natural desire is to formulate a gauge-invariant definition of the TLN. However, this is not straightforward, as infinitely many gauge-invariant quantities can be constructed once a gauge-fixed expression is chosen. An alternative strategy is to work within a well-suited gauge. The optimal gauge depends on the specific context of measurement. For instance, in cosmology, the halo bias parameter is most naturally defined in synchronous coordinates, which are commonly used in the spherical collapse model [50]. The challenge with the static TLN is that it is not measured directly but instead inferred from Bayesian analysis based on a model-dependent waveform fit.

On the other hand, an important advantage of our method is its non-perturbative nature, as we have found an exact solution to the full Einstein equations for a vacuum static and axisymmetric spacetime. The vanishing of the Love number then is derived by analyzing the behavior of the Kretschmann scalar near the event horizon. Since the Kretschmann scalar encapsulates intrinsic curvature properties, this conclusion is ultimately independent of the coordinate choice.

The paper is organized as follows: Section 2 reviews the Weyl class of static, axisymmetric vacuum solutions and introduces the Ernst potential formalism. Section 3 revisits the tidal response of Schwarzschild BHs, establishing the framework for non-linear tidal effects. Section 4 extends the analysis to Kerr BHs, detailing the transition to prolate spheroidal coordinates and examining the decaying and growing quadrupole modes. Section 5 investigates the impact of non-linear tidal interactions and their role in ensuring the vanishing of TLNs. Section 6 discusses the role played by the non-linear symmetries. Section 7 concludes with a discussion of the implications and potential extensions of this work. Finally, Appendices A and B discuss the transition to Boyer-Lindquist coordinates and other multipole basis, offering a complementary perspective.

6.2 The Weyl class of static, axisymmetric vacuum solutions

As demonstrated by Ernst [18], the field equations for a uniformly rotating, axially symmetric source can be reformulated using a simple variational principle. Following this approach unified solutions for Weyl and Papapetrou metrics emerge providing us with a direct derivation of the Schwarzschild as well as the Kerr metric in prolate spheroidal coordinates. New solutions for the case of Kerr BH in tidal environments can also be obtained in this way, allowing us to make statements about the non-linear static love numbers of Kerr BHs. We can start our analysis by considering a static axisymmetric Weyl metric in the following form [40]

$$\mathrm{d}s^2 = f^{-1}\left[e^{2\gamma}(\mathrm{d}\rho^2 + \mathrm{d}z^2) + \rho^2\mathrm{d}\varphi^2\right] - f(\mathrm{d}t - \omega\mathrm{d}\varphi)^2, \tag{6.1}$$

where $f = f(\rho, z)$, $\omega = \omega(\rho, z)$ and $\gamma = \gamma(\rho, z)$.[1] It turns out that that the equations for f and ω which follow from the vacuum Einstein field equations ($R_{\mu\nu}$=0) can be decoupled from the equation for $\gamma(\rho, z)$ and are given by

$$f\nabla^2 f = \nabla f \cdot \nabla f - \rho^{-2} f^4 \nabla\omega \cdot \nabla\omega, \tag{6.2}$$

$$\nabla \cdot \left(\rho^{-2} f^2 \nabla\omega\right) = 0. \tag{6.3}$$

[1] Let us note that for $\rho \to 0$ we should have $\gamma \to 0$, since otherwise the metric would contain a part proportional to $e^{2\gamma(0,z)}\mathrm{d}\rho^2 + \rho^2\mathrm{d}\varphi^2$, which clearly has a conical singularity for any z.

We may now introduce a new scalar ϕ from ω as

$$\nabla\phi = -\frac{f^2}{\rho}\hat{n}_\varphi \times \nabla\omega \tag{6.4}$$

where $\hat{n}_\varphi$ is the unit vector in the φ direction. It has been shown [18] that Eqs. (6.2) and (6.3) can also be obtained through a complex function, the Ernst potential $\mathcal{E}$, defined as

$$\mathcal{E} = f + i\phi. \tag{6.5}$$

Moreover, the equations for the third function $\gamma(r,\theta)$ are written in terms of the $\mathcal{E}$ as [19]

$$\begin{aligned}
\gamma_{,z} &= \frac{1}{4}\rho f^{-2}\left[(\mathcal{E}_{,\rho})(\mathcal{E}^*_{,z}) + (\mathcal{E}_{,z})(\mathcal{E}^*_{,\rho})\right], \\
\gamma_{,\rho} &= \frac{1}{4}\rho f^{-2}\left[(\mathcal{E}_{,\rho})(\mathcal{E}^*_{,\rho}) - (\mathcal{E}_{,z})(\mathcal{E}^*_{,z})\right].
\end{aligned} \tag{6.6}$$

We now introduce prolate spheroidal coordinates (t, x, y, φ) instead of Weyl coordinates by writing [52, 44]

$$\begin{aligned}
&\rho = \rho_0(x^2-1)^{1/2}(1-y^2)^{1/2}, \quad x \geq 1, \quad |y| \leq 1, \\
&z = \rho_0 xy, \quad \rho_0 = \text{constant}.
\end{aligned} \tag{6.7}$$

We will see later that ρ_0 is related to the mass and the spin parameter of the BHs we are interested in describing. In such coordinates, the metric in Eq. (6.1) is written as

$$\mathrm{d}s^2 = \rho_0^2 f^{-1}\left[e^{2\gamma}(x^2-y^2)\left(\frac{\mathrm{d}x^2}{x^2-1} + \frac{\mathrm{d}y^2}{1-y^2}\right) + (x^2-1)(1-y^2)\mathrm{d}\varphi^2\right] - f(\mathrm{d}t - \omega\mathrm{d}\varphi)^2. \tag{6.8}$$

Furthermore, for later use, the differential operators we previously introduced are written in prolate spheroidal coordinates now take the following form:

$$\nabla \equiv \rho_0^{-1}(x^2-y^2)^{-1/2}\left[\hat{n}_x(x^2-1)^{1/2}\partial_x + \hat{n}_y(1-y^2)^{1/2}\partial_y\right],$$
$$\nabla^2 = \rho_0^{-2}(x^2-y^2)^{-1}\Big\{\partial_x\Big[(x^2-1)\partial_x\Big] + \partial_y\Big[(1-y^2)\partial_y\Big]\Big\}, \tag{6.9}$$

whereas, the inner product of the gradients of two functions A and B is

$$\nabla A\cdot\nabla B = \rho_0^{-2}(x^2-y^2)^{-1}\Big[(x^2-1)\partial_x A\,\partial_x B + (1-y^2)\partial_y A\,\partial_y B\Big].$$

Equations (6.2) and (6.3), are equivalent to the equation of motion for the Ernst potential $\mathcal{E}$ which are derived from the action

$$\mathcal{S}_{\mathcal{E}} = \int \frac{\nabla\mathcal{E}\cdot\nabla\mathcal{E}^*}{(\mathcal{E}+\mathcal{E}^*)^2} d^2x, \tag{6.10}$$

so that the corresponding equations

$$(\mathcal{E}+\mathcal{E}^*)\,\nabla^2\mathcal{E} - \nabla\mathcal{E}\cdot\nabla\mathcal{E} = 0, \tag{6.11}$$

reproduce Eqs. (6.2) and (6.3). As a result, the problem of finding axisymmetric, stationary vacuum solutions to the Einstein equations is in fact reduced to appropriately solve Eq. (6.11) for the Ernst potential $\mathcal{E}$.

6.3 The Schwarzschild BH in external tidal fields

Although the Schwarzschild BH in external tidal fields has been extensively discussed in Ref. [34], let us recall here its description in terms of the Ernst potential. The latter for the static Schwarzschild metric, with $\omega = 0$ and in prolate spheroidal coordinates, is a real function and it is given by

$$\mathcal{E} = e^{2\psi}\frac{x-1}{x+1}, \tag{6.12}$$

where $\psi(x,y)$ is a real potential. By substituting the above expression into the equation of motion (6.11), we we find that ψ satisfies Laplace equation

$$\nabla^2\psi = 0. \tag{6.13}$$

Let us stress that although (6.11) is a non-linear equation capturing the non-linearily of Einstein equations, (6.13) is an exact linear equation obeyed by ψ. The whole non-linearity has been transmitted to the function γ, which is specified by the non-linear equation (6.6). This is the beauty of the Ernst approach, where a linear equation is separated out of the full non-linear system. Therefore, the solution for $\psi(x,y)$ can then be written as a multipole expansion

$$\psi = \sum_{\ell\geq 1} U_\ell(x)Y_\ell(y)\ , \tag{6.14}$$

where U_ℓ and Y_ℓ satisfy

$$\frac{\mathrm{d}}{\mathrm{dx}}\left((x^2-1)\frac{\mathrm{d}}{\mathrm{dx}}U_\ell\right) - \ell(\ell+1)U_\ell = 0, \tag{6.15}$$

$$\frac{\mathrm{d}}{\mathrm{dy}}\left((1-y^2)\frac{\mathrm{d}}{\mathrm{dy}}Y_\ell\right) + \ell(\ell+1)Y_\ell = 0. \tag{6.16}$$

The regular solution of Eq. (6.16) at $y=\pm 1$ is given by the Legendre polynomials

$$Y_\ell(y) = P_\ell(y), \qquad \ell = 0, 1, \cdots\ , \tag{6.17}$$

and similarly, the solution to Eq. (6.15) is

$$U_\ell = \alpha_\ell \, x^\ell \, {}_2F_1\left(\frac{1-\ell}{2}, -\frac{\ell}{2}, \frac{1-2\ell}{2}, \frac{1}{x^2}\right) + \beta_\ell \frac{1}{x^{\ell+1}} \, {}_2F_1\left(\frac{1+\ell}{2}, \frac{2+\ell}{2}, \frac{3+2\ell}{2}, \frac{1}{x^2}\right), \tag{6.18}$$

so that the function $\psi(x,y)$ turns out to be

$$\psi(x,y) = \sum_{\ell=1}^{\infty}\left[\alpha_\ell \, x^\ell \, {}_2F_1\left(\frac{1-\ell}{2}, -\frac{\ell}{2}, \frac{1-2\ell}{2}, \frac{1}{x^2}\right) + \frac{\beta_\ell}{x^{\ell+1}} \, {}_2F_1\left(\frac{1+\ell}{2}, \frac{2+\ell}{2}, \frac{3+2\ell}{2}, \frac{1}{x^2}\right)\right] P_\ell(y). \tag{6.19}$$

We have seen in Ref. [34], that the decaying mode (proportional to $r^{-\ell-1}$) generates a naked singularity at the horizon $x = 1$.[2] One can verify this by checking the Kretschmann scalar, which diverges for $\beta_\ell \neq 0$ at $x = 1$ as has been shown in [34], where it was also shown that the growing mode (proportional to r^ℓ) is not singular at the horizon. Therefore, $\beta_\ell = 0$ which leads to the vanishing of the static Love number for Schwarzschild BH in an external gravitational field at all orders in the tidal parameter [34].

Let us also note that the solution in Eq. (6.19) determines also the function $\gamma(x,y) = \gamma_s(x,y)$ for the Schwarzschild BH by the equations (6.6), which now are written explicitly in prolate spheroidal coordinates as

$$\begin{aligned}
\gamma_{s,x} &= \frac{1-y^2}{x^2-y^2}\Big[x\left(x^2-1\right)U_{,x}^2 - x\left(1-y^2\right)U_{,y}^2 - 2y\left(x^2-1\right)U_{,x}U_{,y}\Big], \\
\gamma_{s,y} &= \frac{x^2-1}{x^2-y^2}\Big[y\left(x^2-1\right)U_{,x}^2 - y\left(1-y^2\right)U_{,y}^2 + 2x\left(1-y^2\right)U_{,x}U_{,y}\Big],
\end{aligned} \tag{6.20}$$

[2] The hypersurface $x = 1$ is indeed a horizon since the function $f = Re(\mathcal{E})$ in the metric (6.1) vanishes $f(x=1) = 0$. This can also be seen from the definition of the prolate coordinates defined in Eq. (6.7) which cover the region from $x \in [1,\infty)$, i.e., the region outside the black hole horizon.

where

$$U(x,y) = \frac{1}{2}\ln\left(\frac{x-1}{x+1}\right) + \psi(x,y). \tag{6.21}$$

Then, the general solution for $\gamma_s(x,y)$, is provided by the closed formula [52]

$$\gamma_s(x,y) = (x^2-1)\int_{-1}^{y} \frac{\Gamma(x,y')}{x^2-y'^2}\,\mathrm{d}y', \tag{6.22}$$

where

$$\Gamma(x,y) = y(x^2-1)U_{,x}^2 - y(1-y^2)U_{,y}^2 + 2x(1-y^2)U_{,x}U_{,y}\ .$$

6.4 Kerr BH in external tidal fields

In order to introduce rotation, one needs to consider non-zero ω in the metric (6.1). In this case, we expect (6.1) to describe the Kerr BH as well as its embedding in external tidal fields, much the same way as in the non-rotating Schwarzschild background we described in the previous section. Since for a rotating BH ω is not vanishing, the Ernst potential should have a non-zero imaginary part ϕ, which is is determined by Eq. (6.4).

In particular, it has been shown [7, 48] that the correct choice for the Ernst potential for a Kerr BH in an external tidal gravitational field has the form

$$\mathcal{E} = e^{2\psi}\,\frac{x(1+ab)+iy(b-a)-(1-ia)(1-ib)}{x(1+ab)+iy(b-a)+(1-ia)(1-ib)}, \tag{6.23}$$

where $a = a(x,y)$ and $b = b(x,y)$. Then the equations of motion (6.11) for $\mathcal{E}$ turn out to be following equations for $a(x,y)$, $b(x,y)$ and $\psi(x,y)$

$$
\begin{aligned}
\nabla^2\psi &= 0, \\
(x-y)a_{,x} &= 2a\left[(xy-1)\psi_{,x} + (1-y^2)\psi_{,y}\right], \\
(x-y)a_{,y} &= 2a\left[-(x^2-1)\psi_{,x} + (xy-1)\psi_{,y}\right], \\
(x+y)b_{,x} &= -2b\left[(xy+1)\psi_{,x} + (1-y^2)\psi_{,y}\right], \\
(x+y)b_{,y} &= -2b\left[-(x^2-1)\psi_{,x} + (xy+1)\psi_{,y}\right].
\end{aligned}
\tag{6.24}
$$

In addition, Eq. (6.4) is written explicitly as

$$
\begin{aligned}
\phi_{,x} &= \rho_0^{-1}(x^2-1)^{-1} f^2 \omega_{,y}\ , \\
\phi_{,y} &= \rho_0^{-1}(y^2-1)^{-1} f^2 \omega_{,x}.
\end{aligned}
\tag{6.25}
$$

Then, the functions f, γ and ω in the metric (6.1) turn out to be :

$$
\begin{aligned}
f &= e^{2\psi} A B^{-1}, \\
e^{2\gamma} &= K_1 (x^2-1)^{-1} e^{2\gamma_s} A, \\
\omega &= 2\rho_0 e^{-2\psi} A^{-1} C + K_2,
\end{aligned}
\tag{6.26}
$$

where

$$
\begin{aligned}
A &= (x^2-1)(1+ab)^2 - (1-y^2)(b-a)^2, \\
B &= \left[x+1+(x-1)ab\right]^2 + \left[(1+y)a+(1-y)b\right]^2, \\
C &= (x^2-1)(1+ab)\left[b-a-y(a+b)\right] + (1-y^2)(b-a)\left[1+ab+x(1-ab)\right].
\end{aligned}
\tag{6.27}
$$

From the first of Eqs. (6.26) we see that the (tt)-component of the metric is given by $g_{tt} = e^{2\psi} A B^{-1}$ so that $g_{tt} \approx e^{2\psi}$ for $x \to \infty$. In Eq. (6.27), K_1 and K_2 are constants, whereas γ_s is the potential γ of the corresponding static metric given in Eq. (6.22).

6.4.1 The Kerr metric in Weyl coordinates

For $a = b = 0$, the Ernst potential in Eq.(6.23) reduces to the corresponding potential of Eq. (6.12) for the Schwarzschild BH. We will in the following demonstrate, similarly, we can recover the Kerr metric from the potential in Eq. (6.23). This is possible when

$$a = -\alpha, \qquad b = \alpha, \qquad \alpha = \text{const.}. \tag{6.28}$$

In this case we find that [8]

$$\begin{aligned} \mathrm{Re}\{\mathcal{E}\} \equiv f =& \frac{p^2x^2 + q^2y^2 - 1}{(px+1)^2 + q^2y^2}, \\ e^{2\gamma} =& \frac{(px)^2 + (qy)^2 - 1}{p^2(x^2 - y^2)}, \\ \omega =& -2\rho_0 \frac{q\,(px+1)\,(1-y^2)}{p(p^2x^2 + q^2y^2 - 1)}, \end{aligned} \tag{6.29}$$

where,

$$p = \frac{1-\alpha^2}{1+\alpha^2}, \quad q = \frac{2\alpha}{1+\alpha^2}, \quad p^2 + q^2 = 1. \tag{6.30}$$

In the same spirit, we can substitute the imaginary part of Ernst potential in Eq. (6.25) to find ω and by using Eqs. (6.27) with

$$K_1 = \frac{1}{(1-\alpha^2)^2}, \qquad K_2 = -\frac{4\rho_0\alpha}{1-\alpha^2}, \tag{6.31}$$

we end up with Kerr metric in prolate coordinates. The transition to Boyer–Lindquist coordinates can be made by the following set of substitutions

$$\rho_0 x = r - m, \quad y = \cos\theta, \quad \rho_0 = mp, \quad a_0 = mq, \quad \rho_0^2 = m^2 - a_0^2, \tag{6.32}$$

where m is the BH mass and a_0 is the spin parameter of the Kerr BH. In addition, since the spin always satisfy $m^2 \geq a_0^2$, it follows from Eq. (6.32) that the range of α is $|\alpha| \leq 1$. We therefore end up with the known form of the Kerr metric:

$$\begin{aligned} \mathrm{d}s^2 = & -\left(1 - \frac{2mr}{\Sigma}\right)\mathrm{d}t^2 + \frac{\Sigma}{\Delta}\mathrm{d}r^2 + \Sigma \mathrm{d}\theta^2 - \frac{4ma_0 r \sin^2\theta}{\Sigma}\,\mathrm{d}t\,\mathrm{d}\varphi \\ & + \left(r^2 + a_0^2 + \frac{2ma_0^2 r}{\Sigma}\sin^2\theta\right)\sin^2\theta \mathrm{d}\varphi^2 \ , \end{aligned} \tag{6.33}$$

where, as usual,

$$\Delta = r^2 - 2mr + a_0^2, \qquad \Sigma = r^2 + a_0^2\cos^2\theta. \tag{6.34}$$

Therefore, we see that indeed, the Ernst potential (6.23) with a and b as in Eq. (6.28) describes the Kerr metric in prolate spheroidal coordinates.

While Boyer-Lindquist coordinates are preferable in general for describing the Kerr metric some tasks that we encounter later in this paper seem to prefer treatment using Weyl spherical coordinates. Spherical coordinates (R, u, φ) can be expressed in terms of Weyl canonical coordinates (ρ, z, φ) and Boyer-Lindquist coordinates (r, θ, φ) as

$$\begin{aligned} R = & \rho_0\sqrt{x^2 + y^2 - 1} = \sqrt{\rho^2 + z^2} = \sqrt{(r-m)^2 - \rho_0^2\sin^2\theta}\ , \\ \cos u = & \frac{xy}{\sqrt{x^2+y^2-1}} = \frac{z}{\sqrt{\rho^2+z^2}} = \frac{(r-m)\cos\theta}{\sqrt{(r-m)^2 - \rho_0^2\sin^2\theta}}. \end{aligned} \tag{6.35}$$

Transitions between all the previously mentioned coordinate systems can be significantly simplified using the auxiliary functions R_+ and R_- defined as

$$R_\pm(\rho, z) = \sqrt{\rho^2 + (z \pm \rho_0)^2} = (r - m) \pm \rho_0\cos\theta = \sqrt{R^2 + \rho_0^2 \pm 2\rho_0 R\cos u}\ . \tag{6.36}$$

Note that, with the use of (6.32) one obtains quite trivially the inverse transformation of Eq. (6.7) as

$$\begin{aligned}
\rho_0 x &= \frac{1}{2}\left(R_+ + R_-\right) = \frac{1}{2}\left(\sqrt{\rho^2 + (z+\rho_0)^2} + \sqrt{\rho^2 + (z-\rho_0)^2}\right), \\
\rho_0 y &= \frac{1}{2}\left(R_+ - R_-\right) = \frac{1}{2}\left(\sqrt{\rho^2 + (z+\rho_0)^2} - \sqrt{\rho^2 + (z-\rho_0)^2}\right),
\end{aligned} \tag{6.37}$$

In the new coordinates (R, u, φ), the Kerr metric is written as

$$\mathrm{d}s^2 = f^{-1}\left[e^{2\gamma}(\mathrm{d}R^2 + R^2\mathrm{d}u^2) + R^2\sin^2 u\ \mathrm{d}\varphi^2\right] - f(\mathrm{d}t - \omega\mathrm{d}\varphi)^2, \tag{6.38}$$

where

$$\begin{aligned}
f &= 1 - \frac{4m(R_+ + R_- + 2m)}{(R_+ + R_- + 2m)^2 + \frac{a_0^2}{m^2 - a_0^2}(R_+ - R_-)^2}, \\
e^{2\gamma} &= \frac{(R_+ + R_-)^2 - 4m^2 + \frac{a_0^2}{m^2 - a_0^2}(R_+ - R_-)^2}{4R_+ R_-}, \\
\omega &= -\frac{a_0 m(R_+ + R_- + 2m)(4 - \frac{(R_+ - R_-)^2}{(m^2 - a_0^2)})}{(R_+ + R_-)^2 - 4m^2 + a_0^2\frac{(R_+ - R_-)^2}{(m^2 - a_0^2)}}.
\end{aligned} \tag{6.39}$$

Finally, the metric (6.38) can be rewritten in the known form of (6.33) in terms of the Boyer-Lindquist coordinates (r, θ, φ) by the coordinate transformations of (6.35) and (6.36).

6.5 The Kerr BH in external tidal fields

An inspection of Eqs. (6.24) shows that both a and b are determined only up to a multiplicative constant. Therefore, we can utilize this freedom by choosing always the constant value of a and b as in Eq. (6.28). Let us now rewrite the metric in Eq.

(6.8) in the following way

$$\mathrm{d}s^2 = -f\left(\mathrm{d}t - \omega\mathrm{d}\varphi\right)^2 + h\left(\frac{\mathrm{d}x^2}{x^2-1} + \frac{\mathrm{d}y^2}{1-y^2}\right) + \rho_0^2 f^{-1}(x^2-1)(1-y^2)\mathrm{d}\varphi^2 \ , \quad (6.40)$$

where

$$h = \frac{\rho_0^2}{(1-\alpha^2)^2} B e^{-2\psi+2V}, \quad (6.41)$$

$$V = \gamma_s - \frac{1}{2}\ln\left(\frac{x^2-1}{x^2-y^2}\right), \quad (6.42)$$

where the constants $K1$ and K_2 have been fixed by the requirement that the metric (6.40) reduces to the Kerr metric when $\psi = 0$. Form Eqs. (6.24) we see that ψ satisfies the Laplace equation, which in (R, u) coordinates is written as

$$\frac{1}{R^2}\partial_R\Big(R^2\partial_R\psi\Big) + \frac{1}{R^2\sin u}\partial_u\Big(\sin u\partial_u\psi\Big) = 0, \quad (6.43)$$

and thus a general solution in Weyl spherical harmonics must be of the form

$$\psi = \sum_{\ell\geq 1}\left(c_\ell R^\ell + \frac{d_\ell}{R^{\ell+1}}\right) P_\ell(\cos u). \quad (6.44)$$

Let us note that the solution above can also be expressed in prolate coordinates (x, y) by replacing $R = R(x, y)$ and $u = u(x, y)$ as given in Eq. (6.35). We stress again that the solution (6.44) is an exact solution of the exact linear equation (6.13) expressed in (R, u) coordinates, and no approximation is involved in determining Eq. (6.44). Notice that the series starts from $\ell = 1$ since the $\ell = 0$ term yields the Kerr solution and has been factored out in the parametrization of the Ernst potential in Eq. (6.23). It also important to demand the absence of conical singularities. As we have see in section 2, we should have $\lim_{\rho\to 0}\gamma = 0$, which of prolate coordinates is written as $\lim_{y\to\pm 1}\gamma = 0$. Then, as it turns out, conical singularities along the symmetry axis are absent if the following condition is satisfied [51]

$$\sum_{n=0}^{\infty} c_{2n+1} = 0 \ . \tag{6.45}$$

In particular, as shown in [51], condition (6.45) follows from the regularity of the induced metric on the horizon. Indeed, calculating the Euler number χ of the two-dimensional compact 2-surface of the horizon at $x = 1$, it turns out that the condition (6.45) has to be satisfied in order to have $\chi = 2$, i.e., a surface topologically equivalent to a 2-sphere. Therefore, we cannot have for example a single dipole without an octuple tidal deformation.

6.5.1 The decaying quadrupole mode

The solution for ψ in Eq. (6.44) is the sum of decaying modes (proportional to $R^{-\ell-1}$) and growing modes (proportional to R^{ℓ}). Here, we will examine the quadrupole modes, consequently the solution for ψ that we will consider will be of the form:

$$\psi = \left(c_2 R^2 + \frac{d_2}{R^3} \right) P_2(\cos u), \tag{6.46}$$

where c_2 and d_2 are the strength of the growing and decaying tidal fields, respectively. We can now calculate the general expressions for a, b, and V using equations (6.24) and (6.22) for γ_s

$$\begin{aligned} a(x,y) =& -\alpha \exp\Big\{ 2c_2(xy+1)(x-y) - d_2 \Big[\left(x^2+y^2-1\right)^{-5/2} \left(2x^5 + 5x^3\left(y^2-1\right)\right. \\ & \left. -x^2 y\left(5y^2-3\right) - 3x\left(y^2-1\right) - y\left(2y^4-5y^2+3\right)\right) - 2\Big] \Big\}, \end{aligned} \tag{6.47}$$

$$\begin{aligned} b(x,y) =& \alpha \exp\Big\{ 2c_2(1-xy)(x+y) - d_2 \Big[\left(x^2+y^2-1\right)^{-5/2} \left(2x^5 + 5x^3\left(y^2-1\right)\right. \\ & \left. +x^2 y\left(5y^2-3\right) - 3x\left(y^2-1\right) + y\left(2y^4-5y^2+3\right)\right) + 2\Big] \Big\}, \end{aligned} \tag{6.48}$$

$$
\begin{aligned}
V(x,y) =& -\frac{1}{8}\left(y^2-1\right)\Bigg(-2c_2^2x^4\left(9y^2-1\right)+4c_2^2x^2\left(5y^2-1\right)-\frac{24d_2y^2\left(-c_2y^4+c_2y^2+x\right)}{\left(x^2+y^2-1\right)^{5/2}}+\\
&\frac{8d_2\left(x-6c_2y^4\right)}{\left(x^2+y^2-1\right)^{3/2}}+16c_2x+\frac{75d_2^2\left(y^2-1\right)^2y^6}{\left(x^2+y^2-1\right)^6}-\frac{9d_2^2\left(25y^4-38y^2+13\right)y^4}{\left(x^2+y^2-1\right)^5}\\
&+\frac{9d_2^2\left(25y^4-26y^2+5\right)y^2}{\left(x^2+y^2-1\right)^4}-\frac{3d_2^2\left(25y^4-14y^2+1\right)}{\left(x^2+y^2-1\right)^3}\Bigg)+\frac{c_2^2y^4}{4}-\frac{c_2^2y^2}{2}\\
&+\frac{d_2\left(3c_2\left(y^4-1\right)+2x\right)}{\sqrt{x^2+y^2-1}}+V_0.
\end{aligned}
\tag{6.49}
$$

Note that the expressions for $a(x,y)$, $b(x,y)$ and $V(x,y)$ above are exact expressions, and they are the result of integrating (6.24) after using ψ given in Eq. (6.44), where we have substituted $R=R(x,y)$ and $u=u(x,y)$ from Eq. (6.35). The constant V_0 in Eq. (6.49) is determined by the regularity condition [51] $\lim_{y\to\pm1}\gamma(x,y)=0$. Since d_2 in Eq. (6.46) is proportional to the static TLN for quadrupole tidal deformations, we will consider below only the decaying mode. Then, with $c_2=0$, we find that

$$
\psi(x,y) = d_2\frac{1}{R^3}P_2(\cos u), \tag{6.50}
$$

$$
a(x,y) = -\alpha\exp\Big\{-d_2\Big[w(x,y)-w(y,x)\Big]\Big\}, \tag{6.51}
$$

$$
b(x,y) = \alpha\exp\Big\{-d_2\Big[w(x,y)+w(y,x)\Big]\Big\}, \tag{6.52}
$$

$$
\begin{aligned}
A(x,y) =& 4\alpha^2\exp\Big\{-2d_2w(x,y)\Big\}\Big\{(x^2-1)\sinh^2\Big(d_2w(x,y)-\ln|\alpha|\Big)\\
&+(y^2-1)\cosh^2\Big(d_2w(y,x)\Big)\Big\},
\end{aligned}
\tag{6.53}
$$

$$
\begin{aligned}
B(x,y) =& 4\alpha^2\exp\Big\{-2d_2w(x,y)\Big\}\Big\{\Big(\sinh\Big(d_2w(x,y)-\ln|\alpha|\Big)\\
&+\cosh\Big(d_2w(x,y)-\ln|\alpha|\Big)\Big)^2+\Big(y\cosh\Big(d_2w(y,x)\Big)+\sinh\Big(d_2w(y,x)\Big)\Big)^2\Big\},
\end{aligned}
\tag{6.54}
$$

$$
\begin{aligned}
C(x,y) =& 4\,\mathrm{sign}\alpha\,\alpha^2\exp\Big\{-2d_2w(x,y)\Big\}\Big\{(x^2-1)\sinh\Big(d_2w(x,y)-\ln|\alpha|\Big)\Big(\cosh\Big(d_2w(y,x)\Big)\\
&+y\sinh\Big(d_2w(y,x)\Big)\Big)+(1-y^2)\cosh\Big(d_2w(y,x)\Big)\Big(\sinh\Big(d_2w(x,y)-\ln|\alpha|\Big)
\end{aligned}
$$

$$+x \cosh\Big(d_2 w(x,y) - \ln|\alpha|\Big)\Big)\Bigg\}, \tag{6.55}$$

where we have redefined the constant α as αe^{2d_2} and

$$w(x,y) = \frac{l(x,y)}{R^5(x,y)},$$
$$l(x,y) = x\Big(2x^4 + (5x^2-3)(y^2-1)\Big)\,.$$

We are now interested in determining the possible singularities, which in principle can be generated by turning on tidal fields. A way of determining the existence of singularities is by checking if curvature scalars, such as the Kretschmann scalar

$$\mathcal{K} = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}, \tag{6.56}$$

become singular. Since we are discussing the case of the gravitational response of a Kerr BH to an outer gravitational field, we would expect no other singularities other than the known singularities of the Kerr BH. In the opposite case, where new singularities emerge, these should be dressed with a horizon, and they cannot be naked. Therefore, if the Kretschman scalar for example becomes singular somewhere else other than the known Kerr singularities, then the static Love numbers should vanish, provided the new singularities are naked. The complete expression of the Kretschmann scalar is quite long and not at all illuminating. However, expanding at the equator ($y=0$) and close to the outer horizon ($x=1$) of the Kerr BH, we find that the Kretschmann scalar is

$$\mathcal{K}(s,0) \sim \mathcal{K}_0 + e^{\frac{3d_2^2}{16s^3}}\left(\frac{d_2^6}{s^{12}} + \mathcal{O}(s^{-11})\right), \qquad s = x-1, \tag{6.57}$$

where $\mathcal{K}_0$ is the Kretschmann scalar for the Kerr metric at ($x=1, y=0$). We see that if $d_2 \neq 0$, then the Kretschmann scalar becomes singular as $s \to 0$ indicating the appearance of a naked singularity. The singularity should be removed since it is naked. This is achieved by taking $d_2 = 0$, and therefore, the Love number of the Kerr BH vanishes to any order in the tidal field.

One may wonder what would happen if we had kept both c_2 and d_2 non-vanishing. In this case, the Kretschmann scalar $\mathcal{K}$ becomes an extremely lengthy expression, making it difficult to draw definite conclusions. However, since $\mathcal{K}$ is analytic in both c_2 and d_2, it can be expanded in powers of these parameters. This expansion reveals that the term independent of c_2 near $x = 1$ is always given by Eq. (6.57). Additionally, there are finite terms proportional to positive powers of c_2. Consequently, $\mathcal{K}$ will always diverge whenever $d_2 \neq 0$, for any c_2, not necessarily zero.

6.5.2 The growing quadrupole mode

We have seen above that the decaying mode leads to curvature naked singularities and therefore, the TLNs of the Kerr BH should be zero at the full non-linear level. In the following, we will similarly study the growing quadrupole ($\ell = 2$) mode. It has been shown previously [48] that if we keep only growing modes (as we will in our case since TLN's vanish), then analytic expressions for $a\big(R(x,y), u(x,y)\big)$ and $b\big(R(x,y), u(x,y)\big)$ can be calculated, and hence the metric components can be written explicitly in terms of Legendre polynomials for arbitrary ℓ as follows [7]

$$\psi = \sum_{\ell=1}^{\infty} c_\ell \left(\frac{R}{\rho_0}\right)^{\ell} P_\ell(\cos u), \tag{6.58}$$

$$a = -\,\alpha \exp\left\{2\sum_{n=1}^{\infty} c_n \frac{R_-}{\rho_0} \sum_{\ell=0}^{n-1} \left(\frac{R}{\rho_0}\right)^{\ell} P_\ell(\cos u)\right\}, \tag{6.59}$$

$$b = \alpha \exp\left\{2\sum_{n=1}^{\infty} c_n \frac{R_+}{\rho_0} \sum_{\ell=0}^{n-1} (-1)^{n-\ell} \left(\frac{R}{\rho_0}\right)^{\ell} P_\ell(\cos u)\right\}, \tag{6.60}$$

$$V = \sum_{\ell,\ell'=1}^{\infty} \frac{\ell\ell'}{\ell+\ell'} c_\ell c_{\ell'} \left(\frac{R}{\rho_0}\right)^{\ell+\ell'} \left[P_\ell P_{\ell'} - P_{\ell-1} P_{\ell'-1}\right]$$

$$+\sum_{\ell=1}^{\infty} c_\ell \sum_{\ell'=0}^{\ell-1}\left[(-1)^{\ell-\ell'+1}\frac{R_+}{\rho_0}-\frac{R_-}{\rho_0}\right]\left(\frac{R}{\rho_0}\right)^{\ell'} P_{\ell'}, \tag{6.61}$$

$$h =\frac{\rho_0^2}{(1-a^2)^2} B e^{2(V-\psi)}, \tag{6.62}$$

$$\omega =2\rho_0 e^{-2\psi}\frac{C}{A}-\frac{4\rho_0\alpha}{1-\alpha^2}\exp\left(-2\sum_{n=0}^{\infty} c_{2n}\right). \tag{6.63}$$

For the quadrupole $\ell = 2$ deformations we are interested in, the potentials ψ, γ_s and V are then written

$$\psi =c_2\left(\frac{R}{\rho_0}\right)^2 P_2(\cos u), \tag{6.64}$$

$$\begin{aligned}\gamma_s =&\frac{1}{2}\ln\left(\frac{(R_+ + R_-)^2-4\rho_0^2}{4R_+R_-}\right)+c_2^2\left(\frac{R}{\rho_0}\right)^4\left(P_2^2(\cos u)-P_1^2(\cos u)\right)\\ &+c_2\left(\frac{R_+}{\rho_0}\left(\frac{R}{\rho_0}\cos u-1\right)-\frac{R_-}{\rho_0}\left(\frac{R}{\rho_0}\cos u+1\right)\right),\\ V =&c_2^2\left(\frac{R}{\rho_0}\right)^4\left(P_2^2(\cos u)-P_1^2(\cos u)\right)+c_2\left(\frac{R_+}{\rho_0}\left(\frac{R}{\rho_0}\cos u-1\right)-\frac{R_-}{\rho_0}\left(\frac{R}{\rho_0}\cos u+1\right)\right),\end{aligned} \tag{6.65}$$

and therefore, we find

$$f =e^{2\psi}\frac{((R_+ + R_-)^2-4\rho_0^2)(1+ab)^2-(4\rho_0^2-(R_+ - R_-)^2)(b-a)^2}{\left[(R_+ + R_-)(1+ab)+2\rho_0(1-ab)\right]^2+\left[2\rho_0(a+b)+(R_+ - R_-)(a-b)\right]^2}, \tag{6.66}$$

$$f^{-1}e^{2\gamma} =\frac{e^{2(\gamma_s-\psi)}}{(1-a^2)^2}\frac{\left[(R_+ + R_-)(1+ab)+2\rho_0(1-ab)\right]^2+\left[2\rho_0(a+b)+(R_+ - R_-)(a-b)\right]^2}{(R_+ + R_-)^2-4\rho_0^2}, \tag{6.67}$$

$$\begin{aligned}\omega =&e^{-2\psi}\frac{((R_+ + R_-)^2-4\rho_0^2)(1+ab)(2\rho_0(b-a)-(R_+ - R_-)(a+b))}{((R_+ + R_-)^2-4\rho_0^2)(1+ab)^2-(4\rho_0^2-(R_+ - R_-)^2)(b-a)^2}+ \qquad (6.68)\\ &+\frac{(4\rho_0^2-(R_+ - R_-)^2)(b-a)(2\rho_0(1+ab)+(R_+ + R_-)(1-ab))}{((R_+ + R_-)^2-4\rho_0^2)(1+ab)^2-(4\rho_0^2-(R_+ - R_-)^2)(b-a)^2}-\frac{4\rho_0\alpha}{1-\alpha^2}e^{-2c_2},\end{aligned}$$

where

$$
\begin{aligned}
a =& -\alpha \exp\left\{2c_2 \frac{R_-}{\rho_0}\left[1 + \frac{R}{\rho_0}\cos u\right]\right\}, \\
b =& \alpha \exp\left\{2c_2 \frac{R_+}{\rho_0}\left[1 - \frac{R}{\rho_0}\cos u\right]\right\}.
\end{aligned} \tag{6.69}
$$

We should now examine if there are also naked singularities for the growing mode as well. It has been shown in [48], that for $x > 1$ singularities arise whenever $B = 0$, where B has been defined in Eq. (6.27). By assuming that $c_2 < 0$, we find that $B \neq 0$ and therefore, there are no singularities in $x > 1$ in this case. So, the only possibility is to have singularities on the horizon at $x = 1$. Similarly to the decaying mode, the calculation of the Kretschmann scalar $\mathcal{K}(x, y)$ around $x = 1$ shows that

$$
\mathcal{K}(1, y) = \frac{1}{\left(\alpha^2 y^2 + e^{4c_2(y^2-1)}\right)^6} P(y), \tag{6.70}
$$

where $P(y)$ is a polynomial in y. By examining the expression (6.70) analytically we realize that for $c_2 < 0$ and $y \in [-1, 1]$ the denominator is non zero, therefore no singularities arise at the outer horizon of the BH. The behavior (6.70) around $x = 1$, is in accordance with the plots of the Kretschmann scalar given in [2]. It is also important to highlight that the physical consequence of $c_2 < 0$ is that the Kerr BH slows down its rotation when tidal fields are present. This can be formally understood by calculating the angular velocity at the horizon of the BH. One obtains the angular velocity expression in Boyer-Lindquist coordinates as [49]

$$
\Omega_H = -\frac{g_{tt}}{g_{t\phi}}\bigg|_H . \tag{6.71}
$$

By substituting our findings for the quadrupole in (6.71) we obtain

$$
\Omega_H = \frac{a_0}{a_0^2 + r_+^2} e^{2c_2} = \Omega_H^K e^{2c_2}, \tag{6.72}
$$

where $\Omega_H^K = a_0/a_0^2 + r_+^2$ is the angular velocity of the horizon of the Kerr metric. Therefore, as Eq. (6.72) indicates, a Kerr BH would spin up for $c_2 > 0$, which is not physically plausible. In reality, due to tidal braking, a Kerr BH should slow down its rotation, leading to a reduction in its angular momentum when subjected to a tidal field. This is the tidal locking effect which leading to the damping of rotation in binary systems [22]. This behavior is consistent only if $c_2 < 0$, which also explains the emergence of singularities for $c_2 > 0$.

6.6 The role of symmetries

We have seen above that the static tidal Love numbers vanish identically at the full non-linear level not only for a non-rotating BH [34, 13], but also for rotating BHs, suggesting that there is an underlying non-linear symmetry explaining such a behavior also in the case of rotating spacetimes. Such a symmetry already appears at the linear level in the tidal force [25, 10, 9, 26, 27, 11, 28, 30, 6, 32, 3, 4, 17, 45]. In fact, it turns out that for each mode ℓ solving these equations, a conserved quantity P_ℓ exists which is associated with the aforementioned underlying symmetry. The corresponding conserved charges allow for descending to the monopole case ($\ell = 0$) using ladder operators. Conservation of P_0 implies the invariance of P_ℓ for higher modes, providing a framework to understand why the decaying solution $\sim 1/r^{\ell+1}$ must be excluded, as it is tied to divergences at the horizon. The non-linear version of the symmetry has been identified in Refs. [34, 13].

Now, a pivotal observation is that the equation for ψ which governs the static configuration even in the full non-linear regime, retains a linear structure as it solves the Laplace equation. Remarkably, this equation coincides with the one solved in the linear case for a static, massless scalar field in the Schwarzschild background. However, the non-linearities here are encoded in the function $a(x, y)$ and $b(x, y)$, which enter the parametrization of the Ernst potential in Eq. (6.23). Expanding $\psi(x, y)$ as [34]

$$\psi(x, y) = \sum_{\ell=0} U_\ell(x) P_\ell(y), \tag{6.73}$$

we can define the following ladder operators as

$$\begin{aligned} L_\ell^+ &= -(x^2-1)\frac{\mathrm{d}}{\mathrm{dx}} - (\ell+1)x, \\ L_\ell^- &= (x^2-1)\frac{\mathrm{d}}{\mathrm{dx}} - \ell x. \end{aligned} \tag{6.74}$$

These operators act as raising and lowering operators for the multipole moments, satisfying

$$L_\ell^+ U_\ell \sim U_{\ell+1}, \quad L_\ell^- U_\ell \sim U_{\ell-1}. \tag{6.75}$$

Notice that the ladder operator structure applies only to U_ℓ and not to the full Ernst potential $\mathcal{E}$, since the latter does not satisfy a linear equation. Then following the standard constructions used in linear perturbation theory [26], one can define conserved quantities

$$Q_\ell = (x^2-1)\frac{\mathrm{d}}{\mathrm{dx}}\left(L_1^- L_2^- \cdots L_\ell^-\right) U_\ell, \tag{6.76}$$

for which:

$$\frac{\mathrm{dQ}_\ell}{\mathrm{dx}} = 0. \tag{6.77}$$

For the decaying solution, we find that at large x

$$U_\ell \sim \frac{\beta_\ell}{x^{\ell+1}}, \tag{6.78}$$

resulting in a conserved Q_ℓ that remains finite but non-zero as $x \to \infty$. Near the event horizon, this decaying mode diverges logarithmically, as $\ln(x-1)$. Since the growing and decaying modes must share the same Q_ℓ, and the growing mode at the horizon is constant (implying $Q_\ell = 0$), the conservation of Q_ℓ necessitates the exclusion of the decaying solution due to its divergence. However, an additional argument is required. The reason is that, due to the aforementioned divergence, linear perturbation theory breaks down, and one has to consider the full non-linear problem. We found here that indeed the divergence of the decaying mode at the horizon survives at the full non-linear lever and shows off as a naked singularity as the Kretschmann scalar indicates. Therefore, the decaying mode should be completely eliminated leading to a vanishing static Love number. By discarding the decaying modes in $\psi(x, y)$, no extra divergences propagate into the non-linear Ernst potential, ensuring consistency with the Kerr background.

The Laplacian equation satisfied by ψ in Eq. (6.24) is structurally equivalent to that in a two-dimensional flat spacetime in the original Weyl coordinates (ρ, z). Its solutions can therefore be expressed in terms of holomorphic functions

$$\psi(\zeta, \bar{\zeta}) = \Psi(\zeta) + \bar{\Psi}(\bar{\zeta}), \tag{6.79}$$

where $\zeta = \rho + iz$. Any analytic transformation of ζ yields a new solution. Then the ladder operators are generators of a conformal symmetry group associated with these holomorphic transformations.

We should note however, that the above conformal (homolorphic) symmetries are tied up to the symmetries of the Ernst potential $\mathcal{E}$. An inspection of the action (6.10) or of Eq. (6.11) reveals that they are both invariant under the $\mathrm{SL}(2, \mathbb{R})$ group which act on the Ernst potential as

$$\mathcal{E} \to \mathcal{E}' = -i\frac{ai\mathcal{E} + b}{ci\mathcal{E} + d}, \qquad ad - bc = 1, \tag{6.80}$$

or in terms of f and ϕ

$$\begin{aligned} \phi \to \phi' =& -\frac{acf^2 + (d - c\phi)(b - a\phi)}{c^2 f^2 + (d - c\phi)^2} \\ f \to f' =& \frac{f}{c^2 f^2 + (d - c\phi)^2}. \end{aligned} \tag{6.81}$$

The action (6.10) and the equations (6.11) are identical to the action and the equations of motion of a non-linear $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ σ-model in two dimensions. This can be seen from the parametrization of the $\mathrm{SL}(2,\mathbb{R})$ group by the 2×2 matrices

$$V = \begin{pmatrix} V_-^1 & V_+^1 \\ V_-^2 & V_+^2 \end{pmatrix} = \frac{i}{\sqrt{-2if}} \begin{pmatrix} -\mathcal{E}^* e^{-i\vartheta} & \mathcal{E} e^{i\vartheta} \\ \mathcal{E}^* e^{-i\vartheta} & \mathcal{E}^* e^{i\vartheta} \end{pmatrix}. \tag{6.82}$$

There is a local U(1) which is realized by the shifts $\vartheta \to \vartheta + \Delta\vartheta$, and a global SL $(2,\mathbb{R})$ that acts from the left. Clearly then, $\mathcal{E}$ parameterizes the $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ coset space once the local U(1) is fixed. Such non-linear σ-models appear frequently in GR and are often referred to as Ernst models. Originally introduced in the context of Geroch's reduction of GR [21] and extensively studied by Ernst [18], these models provide a framework to understand the symmetry properties of stationary solutions in GR.

In fact, the SL(2,$\mathbb{R}$) symmetry of the Ernst model, due to mixing with the larger conformal (holomorphic) transformations give rise to an infinite algebra, the SL(2,$\mathbb{R}$) infinite dimensional current algebra. The ladder operators stemming out from the Laplace equation are indeed part of the generators of this infinite-dimensional group of transformations, which therefore explain the vanishing of the static TLN for four-dimensional BHs.

This symmetry structure is a hallmark of stationary and axisymmetric space-times, which are inherently linked to the two-dimensional nature of the equations governing such systems. The infinite-dimensional symmetry described above governs the solution space of stationary, axisymmetric spacetimes in Einstein's vacuum field equations. The Ernst models in two-dimensions and the associated symmetry structures have been widely used in studying BH solutions, including the generation of exact solutions such as the Kerr metric or multi-BH configurations. They are also crucial in exploring extensions of general relativity, where similar two-dimensional dynamics occur.

All of the above underscore the rich symmetry structure inherent in the two-dimensional reduction of general relativity. This structure, exemplified by the infinite-dimensional SL $(2, \mathbb{R})$ algebra, provides a powerful tool for understanding stationary, axisymmetric solutions and underlines the vanishing of the static tidal Love numbers [34, 13].

6.7 Conclusions

In this paper, we have analyzed the non-linear tidal response of Kerr BHs under the influence of external gravitational fields. Using the Ernst formalism and Weyl coordinates, we systematically extended previous results for Schwarzschild BHs to the case of rotating Kerr BHs. Our primary finding is that the static tidal Love numbers of Kerr BHs vanish at all orders in the external tidal field, consistent with the unique symmetries and characteristics of these spacetimes.

The vanishing of the static Love numbers reflects the absence of internal structure in BHs and the profound influence of their underlying spacetime symmetries. Unlike neutron stars, which exhibit nonzero Love numbers that depend on their internal composition, BHs are characterized by their event horizons and the no-hair theorem. This result implies that Kerr BHs cannot sustain any multipole deformations in response to external tidal forces, even when higher-order non-linear effects are taken into account. It also emphasizes the resilience of BH spacetimes against tidal perturbations, a property that distinguishes them from other compact objects.

Our analysis highlighted the utility of the Ernst potential in describing the behavior of BHs in tidal environments. By expressing the Kerr metric in Weyl coordinates, we were able to generalize the Schwarzschild case and examine the role of rotational effects. The use of prolate spheroidal coordinates further facilitated the derivation of key results, enabling a rigorous examination of both growing and decaying quadrupole modes. The identification of singularities in the Kretschmann scalar associated with the decaying mode underscores the physical consistency of setting the Love numbers to zero. This approach reaffirms that any tidal-induced singularity must remain hidden behind a horizon, preserving the integrity of the spacetime.

From an astrophysical perspective, the vanishing of Kerr BH Love numbers has significant implications for gravitational wave astronomy. The tidal deformability of BHs is a critical parameter in the modeling of waveforms from binary inspirals, particularly in scenarios involving BH-neutron star or BH-BH mergers. The lack of tidal signatures from BHs simplifies waveform modeling while providing a stringent test of general relativity in the strong-field regime. Furthermore, these results help refine the theoretical foundations for interpreting gravitational wave data, ensuring that deviations from predicted signals are not misattributed to unmodeled BH tidal effects.

Future research could explore several extensions of this work. One avenue is the inclusion of dynamical tidal effects, where time-dependent perturbations may lead to dissipative phenomena or resonances. Another is the investigation of quantum corrections to the Love numbers, particularly in contexts where semiclassical gravity or string theory might introduce additional structure to the spacetime. The study of tidal effects in higher-dimensional BHs or alternative theories of gravity could provide a broader context for understanding the universality of our findings.

Finally, let us note that our solution describes static tides and not tidal dissipation. The latter cannot be captured by our exact solution. The reason is that tidal dissipation is proportional to $(\omega - m\Omega_H)$, where ω is the frequency of the perturbation, m is the “magnetic" quantum number and Ω_H is the angular velocity at the horizon. Since our solution is static ($\omega = 0$), and axisymmetric (no φ-dependence, i.e., $m = 0$), it is clear that it does not describe tidal dissipation.

In conclusion, our results reinforce the fundamental nature of BHs as geometrically simple yet profoundly enigmatic objects. The vanishing of their tidal Love numbers, even in the non-linear regime, exemplifies their remarkable symmetry and resistance to external perturbations. These findings contribute to the deeper understanding of BH physics and its pivotal role in testing the limits of general relativity.

A.K. acknowledges support from the Swiss National Science Foundation (project number IZSEZ0_229414). A.R. acknowledges support from the Swiss National Science Foundation (project number CRSII5_213497) and by the Boninchi Foundation for the project “PBHs in the Era of GW Astronomy”.

Bibliography

Bibliography

[1] R. Abbott et al. Tests of General Relativity with GWTC-3. 12 2021.

[2] S. Abdolrahimi, J. Kunz, P. Nedkova, and C. Tzounis. Properties of the distorted Kerr black hole. *JCAP*, 12:009, 2015.

[3] J. Ben Achour, E. R. Livine, S. Mukohyama, and J.-P. Uzan. Hidden symmetry of the static response of black holes: applications to Love numbers. *JHEP*, 07:112, 2022.

[4] R. Berens, L. Hui, and Z. Sun. Ladder symmetries of black holes and de Sitter space: love numbers and quasinormal modes. *JCAP*, 06:056, 2023.

[5] T. Binnington and E. Poisson. Relativistic theory of tidal Love numbers. *Phys. Rev. D*, 80:084018, 2009.

[6] G. Bonelli, C. Iossa, D. P. Lichtig, and A. Tanzini. Exact solution of Kerr black hole perturbations via CFT2 and instanton counting: Greybody factor, quasinormal modes, and Love numbers. *Phys. Rev. D*, 105(4):044047, 2022.

[7] N. Bretón, T. E. Denisova, and V. S. Manko. A kerr black hole in the external gravitational field. *Physics Letters A*, 230(1):7–11, 1997.

[8] J. Castejon-Amenedo and V. S. Manko. Superposition of the Kerr metric with the generalized Erez-Rosen solution. *Phys. Rev. D*, 41:2018–2020, 1990.

[9] P. Charalambous, S. Dubovsky, and M. M. Ivanov. Hidden Symmetry of Vanishing Love Numbers. *Phys. Rev. Lett.*, 127(10):101101, 2021.

[10] P. Charalambous, S. Dubovsky, and M. M. Ivanov. On the Vanishing of Love Numbers for Kerr Black Holes. *JHEP*, 05:038, 2021.

[11] P. Charalambous, S. Dubovsky, and M. M. Ivanov. Love symmetry. *JHEP*, 10:175, 2022.

[12] H. S. Chia. Tidal deformation and dissipation of rotating black holes. *Phys. Rev. D*, 104(2):024013, 2021.

[13] O. Combaluzier-Szteinsznaider, L. Hui, L. Santoni, A. R. Solomon, and S. S. C. Wong. Symmetries of Vanishing Nonlinear Love Numbers of Schwarzschild Black Holes. 10 2024.

[14] F. Crescimbeni, G. Franciolini, P. Pani, and A. Riotto. Primordial black holes or else? Tidal tests on subsolar mass gravitational-wave observations. 2 2024.

[15] T. Damour and O. M. Lecian. On the gravitational polarizability of black holes. *Phys. Rev. D*, 80:044017, 2009.

[16] T. Damour and A. Nagar. Relativistic tidal properties of neutron stars. *Phys. Rev. D*, 80:084035, 2009.

[17] V. De Luca, J. Khoury, and S. S. C. Wong. Nonlinearities in the tidal Love numbers of black holes. *Phys. Rev. D*, 108(2):024048, 2023.

[18] F. J. Ernst. New formulation of the axially symmetric gravitational field problem. *Phys. Rev.*, 167:1175–1179, 1968.

[19] F. J. Ernst, V. S. Manko, and E. Ruiz. Equatorial symmetry / antisymmetry of stationary axisymmetric electrovac spacetimes. *Class. Quant. Grav.*, 23:4945–4952, 2006.

[20] E. E. Flanagan and T. Hinderer. Constraining neutron star tidal Love numbers with gravitational wave detectors. *Phys. Rev. D*, 77:021502, 2008.

[21] R. P. Geroch. A Method for generating new solutions of Einstein's equation. 2. *J. Math. Phys.*, 13:394–404, 1972.

[22] B. Gladman, D. D. Quinn, P. Nicholson, and R. Rand. Synchronous Locking of Tidally Evolving Satellites. *Icarus*, 122:166–192, 1996.

[23] S. E. Gralla. On the Ambiguity in Relativistic Tidal Deformability. *Class. Quant. Grav.*, 35(8):085002, 2018.

[24] T. Hinderer. Tidal Love numbers of neutron stars. *Astrophys. J.*, 677:1216–1220, 2008. [Erratum: Astrophys.J. 697, 964 (2009)].

[25] L. Hui, A. Joyce, R. Penco, L. Santoni, and A. R. Solomon. Static response and Love numbers of Schwarzschild black holes. *JCAP*, 04:052, 2021.

[26] L. Hui, A. Joyce, R. Penco, L. Santoni, and A. R. Solomon. Ladder symmetries of black holes. Implications for love numbers and no-hair theorems. *JCAP*, 01(01):032, 2022.

[27] L. Hui, A. Joyce, R. Penco, L. Santoni, and A. R. Solomon. Near-zone symmetries of Kerr black holes. *JHEP*, 09:049, 2022.

[28] M. M. Ivanov and Z. Zhou. Vanishing of Black Hole Tidal Love Numbers from Scattering Amplitudes. *Phys. Rev. Lett.*, 130(9):091403, 2023.

[29] T. Katagiri, T. Ikeda, and V. Cardoso. Parametrized Love numbers of nonrotating black holes. *Phys. Rev. D*, 109(4):044067, 2024.

[30] T. Katagiri, M. Kimura, H. Nakano, and K. Omukai. Vanishing Love numbers of black holes in general relativity: From spacetime conformal symmetry of a two-dimensional reduced geometry. *Phys. Rev. D*, 107(12):124030, 2023.

[31] T. Katagiri, K. Yagi, and V. Cardoso. On relativistic dynamical tides: subtleties and calibration. 9 2024.

[32] A. Kehagias, D. Perrone, and A. Riotto. Quasinormal modes and Love numbers of Kerr black holes from AdS_2 black holes. *JCAP*, 01:035, 2023.

[33] A. Kehagias, D. Perrone, and A. Riotto. A Short Note on the Love Number of Extremal Reissner-Nordstrom and Kerr-Newman Black Holes. 6 2024.

[34] A. Kehagias and A. Riotto. Black Holes in a Gravitational Field: The Non-linear Static Love Number of Schwarzschild Black Holes Vanishes. 10 2024.

[35] A. Kehagias and A. Riotto. Can We Detect Deviations from Einstein's Gravity in Black Hole Ringdowns? 11 2024.

[36] A. Le Tiec and M. Casals. Spinning Black Holes Fall in Love. *Phys. Rev. Lett.*, 126(13):131102, 2021.

[37] A. Le Tiec, M. Casals, and E. Franzin. Tidal Love Numbers of Kerr Black Holes. *Phys. Rev. D*, 103(8):084021, 2021.

[38] P. Pani, L. Gualtieri, and V. Ferrari. Tidal Love numbers of a slowly spinning neutron star. *Phys. Rev. D*, 92(12):124003, 2015.

[39] P. Pani, L. Gualtieri, A. Maselli, and V. Ferrari. Tidal deformations of a spinning compact object. *Phys. Rev. D*, 92(2):024010, 2015.

[40] A. Papapetrou. Eine rotationssymmetrische losung in der allgemeinen relativitatstheorie. *Annals Phys.*, 12:309–315, 1953.

[41] E. Poisson. Tidally induced multipole moments of a nonrotating black hole vanish to all post-Newtonian orders. *Phys. Rev. D*, 104(10):104062, 2021.

[42] E. Poisson and C. M. Will. *Gravity: Newtonian, Post-Newtonian, Relativistic.* Cambridge University Press, 2014.

[43] R. A. Porto. The Tune of Love and the Nature(ness) of Spacetime. *Fortsch. Phys.*, 64(10):723–729, 2016.

[44] H. Quevedo. General static axisymmetric solution of Einstein's vacuum field equations in prolate spheroidal coordinates. *Phys. Rev. D*, 39(10):2904, 1989.

[45] M. Rai and L. Santoni. Ladder Symmetries and Love Numbers of Reissner–Nordström Black Holes. 4 2024.

[46] M. M. Riva, L. Santoni, N. Savić, and F. Vernizzi. Vanishing of Quadratic Love Numbers of Schwarzschild Black Holes.

[47] M. M. Riva, L. Santoni, N. Savić, and F. Vernizzi. Vanishing of nonlinear tidal Love numbers of Schwarzschild black holes. *Phys. Lett. B*, 854:138710, 2024.

[48] A. Tomimatsu. Distorted rotating black holes. *Physics Letters A*, 103(8):374–376, 1984.

[49] P. K. Townsend. Black holes: Lecture notes. 7 1997.

[50] D. Wands and A. Slosar. Scale-dependent bias from primordial non-Gaussianity in general relativity. *Phys. Rev. D*, 79:123507, 2009.

[51] B. C. Xanthopoulos. Local toroidal black holes that are static and axisymmetric. *Proceedings of the Royal Society of London. A. Mathematical and Physical Sciences*, 388(1794):117–131, 1983.

[52] D. M. Zipoy. Topology of Some Spheroidal Metrics. *J. Math. Phys.*, 7(6):1137, 1966.